CHAPTER 8

Right Triangle Trigonometry

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next generation textbooks

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CHAPTER 1 Right Triangle Trigonometry

CHAPTER OUTLINE

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1.2 The Pythagorean Theorem
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1.7 Tangent, Sine and Cosine
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1.13 Extension: Laws of Sines and Cosines

Chapter 8 explores right triangles in far more depth than Chapters 4 and 5. Recall that a right triangle is a triangle with exactly one right angle. In this chapter, we will first prove the Pythagorean Theorem and its converse, followed by analyzing the sides of certain types of triangles. Then, we will introduce trigonometry, which starts with the tangent, sine and cosine ratios. Finally, we will extend sine and cosine to any triangle, through the Law of Sines and the Law of Cosines.

1.1 Simplifying Square Roots

Here you'll simplify, add, subtract and multiply square roots.

The length of the two legs of a right triangle are $2\sqrt{5}$ and $3\sqrt{4}$. What is the length of the triangle's hypotenuse?

Guidance

Before we can solve a quadratic equation using square roots, we need to review how to simplify, add, subtract, and multiply them. Recall that the **square root** is a number that, when multiplied by itself, produces another number. 4 is the square root of 16, for example. -4 is also the square root of 16 because $(-4)^2 = 16$. The symbol for square root is the **radical** sign, or $\sqrt{}$. The number under the radical is called the **radicand**. If the square root of a number is not an integer, it is an irrational number.

Example A

Find $\sqrt{50}$ using:

a) A calculator.

b) By simplifying the square root.

Solution:

a) To plug the square root into your graphing calculator, typically there is a $\sqrt{}$ or SQRT button. Depending on your model, you may have to enter 50 before or after the square root button. Either way, your answer should be $\sqrt{50} = 7.071067811865\ldots$ In general, we will round to the hundredths place, so 7.07 is sufficient.

b) To simplify the square root, the square numbers must be "pulled out." Look for factors of 50 that are square numbers: 4, 9, 16, 25... 25 is a factor of 50, so break the factors apart.

$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$. *This is the most accurate answer.*

Radical Rules

1. $\sqrt{ab} = \sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ Any two radicals can be multiplied together.

2. $x\sqrt{a} \pm y\sqrt{a} = x \pm y\sqrt{a}$ The radicands must be the same in order to add or subtract.

3. $\left(\sqrt{a}\right)^2 = \sqrt{a^2} = a$ The square and square root cancel each other out.

Example B

Simplify $\sqrt{45} + \sqrt{80} - 2\sqrt{5}$.

Solution: At first glance, it does not look like we can simplify this. But, we can simplify each radical by pulling out the perfect squares.

$$\sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5}$$
$$\sqrt{80} = \sqrt{16 \cdot 5} = 4\sqrt{5}$$

Rewriting our expression, we have: $3\sqrt{5}+4\sqrt{5}-2\sqrt{5}$ and all the radicands are the same. Using the Order of Operations, our answer is $5\sqrt{5}$.

Example C

Simplify $2\sqrt{35}\cdot 4\sqrt{7}$.

Solution: Multiply across.

$$2\sqrt{35}\cdot 4\sqrt{7} = 2\cdot 4\sqrt{35\cdot 7} = 8\sqrt{245}$$

Now, simplify the radical. $8\sqrt{245} = 8\sqrt{49\cdot 5} = 8\cdot 7\sqrt{5} = 56\sqrt{5}$

Intro Problem Revisit We must use the Pythagorean Theorem, which states that the square of one leg of a right triangle plus the square of the other leg equals the square of the hypotenuse.

So we are looking for c such that $(2\sqrt{5})^2+(3\sqrt{4})^2 = c^2$.

Simplifying, we get $4\cdot 5+9\cdot 4 = c^2$, or $20+36 = c^2$.

Therefore, $c^2 = 56$, so to find c, we must take the square root of 56.

$\sqrt{56} = \sqrt{4\cdot 14} = 2\sqrt{14}$.

Therefore, $c = 2\sqrt{14}$.

Guided Practice

Simplify the following radicals.

1. $\sqrt{150}$
2. $2\sqrt{3}-\sqrt{6}+\sqrt{96}$
3. $\sqrt{8}\cdot\sqrt{20}$

Answers

1. Pull out all the square numbers.

$$\sqrt{150} = \sqrt{25\cdot 6} = 5\sqrt{6}$$

Alternate Method: Write out the prime factorization of 150.

$$\sqrt{150} = \sqrt{2\cdot 3\cdot 5\cdot 5}$$

Now, pull out any number that has a pair. Write it *once* in front of the radical and multiply together what is left over under the radical.

$$\sqrt{150} = \sqrt{2\cdot 3\cdot 5\cdot 5} = 5\sqrt{6}$$

2. Simplify $\sqrt{96}$ to see if anything can be combined. We will use the alternate method above.

$$\sqrt{96} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3} = 2 \cdot 2\sqrt{6} = 4\sqrt{6}$$

Rewrite the expression: $2\sqrt{3} - \sqrt{6} + 4\sqrt{6} = 2\sqrt{3} + 3\sqrt{6}$. This is fully simplified. $\sqrt{3}$ and $\sqrt{6}$ cannot be combined because they do not have the same value under the radical.

3. This problem can be done two different ways.

First Method: Multiply radicals, then simplify the answer.

$$\sqrt{8} \cdot \sqrt{20} = \sqrt{160} = \sqrt{16 \cdot 10} = 4\sqrt{10}$$

Second Method: Simplify radicals, then multiply.

$$\sqrt{8} \cdot \sqrt{20} = \left(\sqrt{4 \cdot 2}\right) \cdot \left(\sqrt{4 \cdot 5}\right) = 2\sqrt{2} \cdot 2\sqrt{5} = 2 \cdot 2\sqrt{2 \cdot 5} = 4\sqrt{10}$$

Depending on the complexity of the problem, either method will work. Pick whichever method you prefer.

Vocabulary

Square Root
A number, that when multiplied by itself, produces another number.

Perfect Square
A number that has an integer for a square root.

Radical
The $\sqrt{}$, or square root, sign.

Radicand
The number under the radical.

Practice

Find the square root of each number by using the calculator. Round your answer to the nearest hundredth.

1. 56
2. 12
3. 92

Simplify the following radicals. If it cannot be simplified further, write "*cannot be simplified*".

4. $\sqrt{18}$
5. $\sqrt{75}$
6. $\sqrt{605}$

7. $\sqrt{48}$
8. $\sqrt{50}\cdot\sqrt{2}$
9. $4\sqrt{3}\cdot\sqrt{21}$
10. $\sqrt{6}\cdot\sqrt{20}$
11. $\left(4\sqrt{5}\right)^2$
12. $\sqrt{24}\cdot\sqrt{27}$
13. $\sqrt{16}+2\sqrt{8}$
14. $\sqrt{28}+\sqrt{7}$
15. $-8\sqrt{3}-\sqrt{12}$
16. $\sqrt{72}-\sqrt{50}$
17. $\sqrt{6}+7\sqrt{6}-\sqrt{54}$
18. $8\sqrt{10}-\sqrt{90}+7\sqrt{5}$

1.2 The Pythagorean Theorem

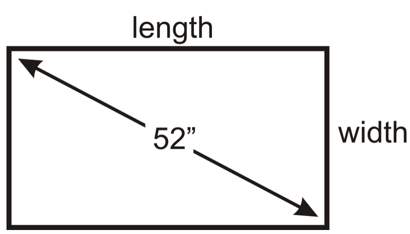
Learning Objectives

- Review simplifying and reducing radicals.
- Prove and use the Pythagorean Theorem.
- Use the Pythagorean Theorem to derive the distance formula.

Review Queue

1. Draw a right scalene triangle.
2. Draw an isosceles right triangle.
3. List all the factors of 75.
4. Write the prime factorization of 75.

Know What? For a 52" TV, 52" is the length of the diagonal. High Definition Televisions (HDTVs) have sides in a ratio of 16:9. What are the length and width of a 52" HDTV?



Simplifying and Reducing Radicals

In algebra, you learned how to simplify radicals. Let's review it here.

Example 1: Simplify the radical.

a) $\sqrt{50}$

b) $\sqrt{27}$

c) $\sqrt{272}$

Solution: For each radical, find the square number(s) that are factors.

a) $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$

b) $\sqrt{27} = \sqrt{9 \cdot 3} = 3\sqrt{3}$

c) $\sqrt{272} = \sqrt{16 \cdot 17} = 4\sqrt{17}$

When adding radicals, you can only combine radicals with the same number underneath it. For example, $2\sqrt{5}+3\sqrt{6}$ cannot be combined, because 5 and 6 are not the same number.

Example 2: Simplify the radicals.

a) $2\sqrt{10}+\sqrt{160}$

b) $5\sqrt{6}\cdot 4\sqrt{18}$

c) $\sqrt{8}\cdot 12\sqrt{2}$

d) $\left(5\sqrt{2}\right)^2$

Solution:

a) Simplify $\sqrt{160}$ before adding: $2\sqrt{10}+\sqrt{160}=2\sqrt{10}+\sqrt{16\cdot 10}=2\sqrt{10}+4\sqrt{10}=6\sqrt{10}$

b) To multiply two radicals, multiply what is under the radicals and what is in front.

$5\sqrt{6}\cdot 4\sqrt{18}=5\cdot 4\sqrt{6\cdot 18}=20\sqrt{108}=20\sqrt{36\cdot 3}=20\cdot 6\sqrt{3}=120\sqrt{3}$

c) $\sqrt{8}\cdot 12\sqrt{2}=12\sqrt{8\cdot 2}=12\sqrt{16}=12\cdot 4=48$

d) $\left(5\sqrt{2}\right)^2=5^2\left(\sqrt{2}\right)^2=25\cdot 2=50\rightarrow$ the $\sqrt{\ }$ and the ^{2} cancel each other out

Lastly, to divide radicals, you need to simplify the denominator, which means multiplying the top and bottom of the fraction by the radical in the denominator.

Example 3: Divide and simplify the radicals.

a) $4\sqrt{6}\div\sqrt{3}$

b) $\frac{\sqrt{30}}{\sqrt{8}}$

c) $\frac{8\sqrt{2}}{6\sqrt{7}}$

Solution: Rewrite all division problems like a fraction.

a)

$$4\sqrt{6}\div\sqrt{3}=\frac{4\sqrt{6}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{4\sqrt{18}}{\sqrt{9}}=\frac{4\sqrt{9\cdot 2}}{3}=\frac{4\cdot 3\sqrt{2}}{3}=4\sqrt{2}$$

like multiplying by 1, $\frac{\sqrt{3}}{\sqrt{3}}$ does not change the value of the fraction

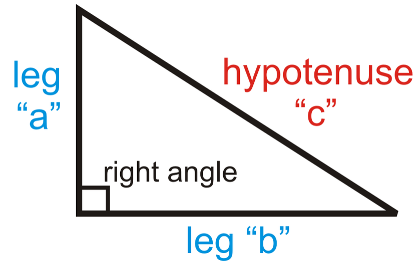
b) $\frac{\sqrt{30}}{\sqrt{8}}\cdot\frac{\sqrt{8}}{\sqrt{8}}=\frac{\sqrt{240}}{\sqrt{64}}=\frac{\sqrt{16\cdot 15}}{8}=\frac{4\sqrt{15}}{8}=\frac{\sqrt{15}}{2}$

c) $\frac{8\sqrt{2}}{6\sqrt{7}}\cdot\frac{\sqrt{7}}{\sqrt{7}}=\frac{8\sqrt{14}}{6\cdot 7}=\frac{4\sqrt{14}}{3\cdot 7}=\frac{4\sqrt{14}}{21}$

Notice, we do not really "divide" radicals, but get them out of the denominator of a fraction.

The Pythagorean Theorem

We have used the Pythagorean Theorem already in this text, but have not proved it. Recall that the sides of a right triangle are the **legs** (the sides of the right angle) and the **hypotenuse** (the side opposite the right angle). For the Pythagorean Theorem, the legs are "a" and "b" and the hypotenuse is "c".

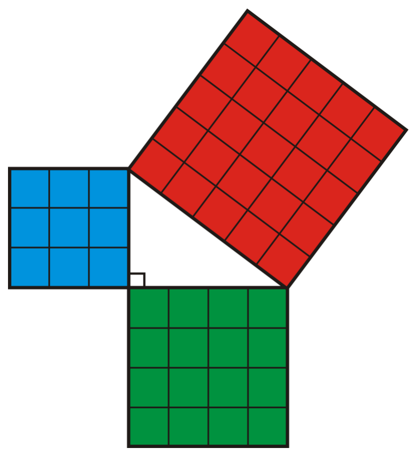


Pythagorean Theorem: Given a right triangle with legs of lengths a and b and a hypotenuse of length c, then $a^2+b^2=c^2$.

Investigation 8-1: Proof of the Pythagorean Theorem

Tools Needed: pencil, 2 pieces of graph paper, ruler, scissors, colored pencils (optional)

1. On the graph paper, draw a 3 in. square, a 4 in. square, a 5 in. square and a right triangle with legs of 3 in. and 4 in.
2. Cut out the triangle and square and arrange them like the picture on the right.



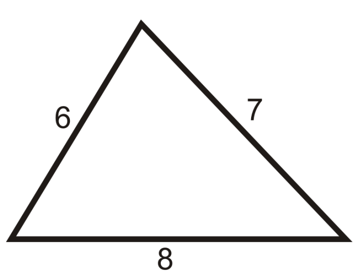
3. This theorem relies on area. Recall that the area of a square is $side^2$. In this case, we have three squares with sides 3 in., 4 in., and 5 in. What is the area of each square?
4. Now, we know that $9+16=25$, or $3^2+4^2=5^2$. Cut the smaller squares to fit into the larger square, thus proving the areas are equal.

For two more proofs, go to: http://www.mathsisfun.com/pythagoras.html and scroll down to “And You Can Prove the Theorem Yourself.”

Using the Pythagorean Theorem

Here are several examples of the Pythagorean Theorem in action.

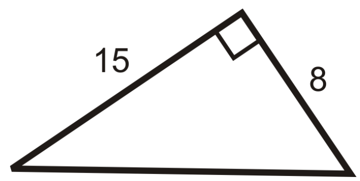
Example 4: Do 6, 7, and 8 make the sides of a right triangle?



Solution: Plug in the three numbers to the Pythagorean Theorem. ***The largest length will always be the hypotenuse.*** If $6^2+7^2=8^2$, then they are the sides of a right triangle.

$$6^2+7^2=36+49=85$$
$$8^2=64 \qquad 85 \neq 64, \text{ so the lengths are not the sides of a right triangle.}$$

Example 5: Find the length of the hypotenuse.

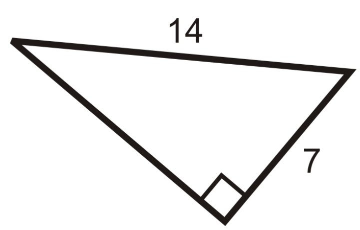


Solution: Use the Pythagorean Theorem. Set $a=8$ and $b=15$. Solve for c.

$$\begin{aligned} 8^2+15^2 &= c^2 \\ 64+225 &= c^2 \\ 289 &= c^2 \qquad \textit{Take the square root of both sides.} \\ 17 &= c \end{aligned}$$

When you take the square root of an equation, the answer is 17 or -17. ***Length is never negative,*** which makes 17 the answer.

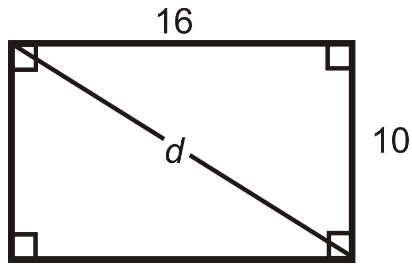
Example 6: Find the missing side of the right triangle below.



Solution: Here, we are given the hypotenuse and a leg. Let's solve for b.

$$\begin{aligned} 7^2+b^2 &= 14^2 \\ 49+b^2 &= 196 \\ b^2 &= 147 \\ b &= \sqrt{147} = \sqrt{49\cdot 3} = 7\sqrt{3} \end{aligned}$$

Example 7: What is the diagonal of a rectangle with sides 10 and 16?



Solution: For any square and rectangle, you can use the Pythagorean Theorem to find the length of a diagonal. Plug in the sides to find d.

$$\begin{aligned} 10^2+16^2 &= d^2 \\ 100+256 &= d^2 \\ 356 &= d^2 \\ d &= \sqrt{356} = 2\sqrt{89} \approx 18.87 \end{aligned}$$

Pythagorean Triples

In Example 5, the sides of the triangle were 8, 15, and 17. This combination of numbers is called a ***Pythagorean triple.***

Pythagorean Triple: A set of three whole numbers that makes the Pythagorean Theorem true.

$$3,4,5 \qquad 5,12,13 \qquad 7,24,25 \qquad 8,15,17 \qquad 9,12,15 \qquad 10,24,26$$

Any multiple of a Pythagorean triple is also considered a triple because it would still be three whole numbers. Multiplying 3, 4, 5 by 2 gives 6, 8, 10, which is another triple. To see if a set of numbers makes a triple, plug them into the Pythagorean Theorem.

Example 8: Is 20, 21, 29 a Pythagorean triple?

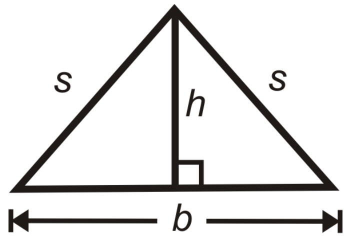
Solution: If $20^2+21^2=29^2$, then the set is a Pythagorean triple.

$$\begin{aligned} 20^2+21^2 &= 400+441 = 841 \\ 29^2 &= 841 \end{aligned}$$

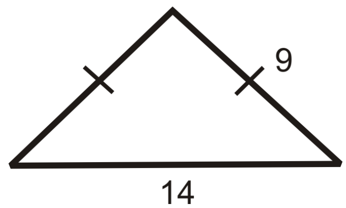
Therefore, 20, 21, and 29 is a Pythagorean triple.

Height of an Isosceles Triangle

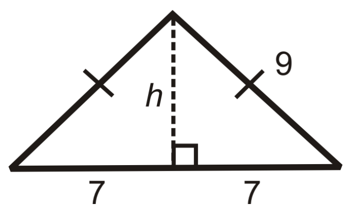
One way to use The Pythagorean Theorem is to find the height of an isosceles triangle.



Example 9: What is the height of the isosceles triangle?



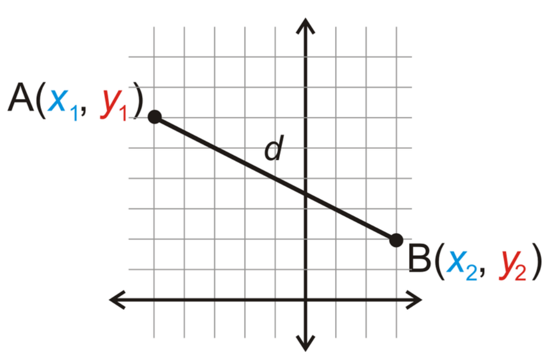
Solution: Draw the altitude from the vertex between the congruent sides, which bisect the base.



$$\begin{aligned}
7^2 + h^2 &= 9^2 \\
49 + h^2 &= 81 \\
h^2 &= 32 \\
h &= \sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}
\end{aligned}$$

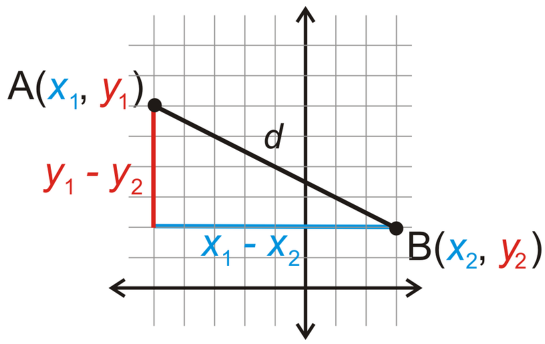
The Distance Formula

Another application of the Pythagorean Theorem is the Distance Formula. We will prove it here.



Let's start with point $A(x_1,y_1)$ and point $B(x_2,y_2)$, to the left. We will call the distance between A and B, d.

Draw the vertical and horizontal lengths to make a right triangle.



Now that we have a right triangle, we can use the Pythagorean Theorem to find the hypotenuse, d.

$$\begin{aligned} d^2 &= (x_1-x_2)^2+(y_1-y_2)^2 \\ d &= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \end{aligned}$$

Distance Formula: The distance $A(x_1,y_1)$ and $B(x_2,y_2)$ is $d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.

Example 10: Find the distance between (1, 5) and (5, 2).

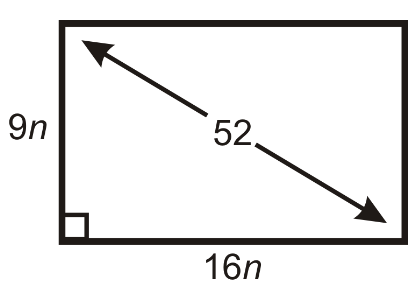
Solution: Make $A(1,5)$ and $B(5,2)$. Plug into the distance formula.

$$\begin{aligned} d &= \sqrt{(1-5)^2+(5-2)^2} \\ &= \sqrt{(-4)^2+(3)^2} \\ &= \sqrt{16+9} = \sqrt{25} = 5 \end{aligned}$$

Just like the lengths of the sides of a triangle, distances are always positive.

Know What? Revisited To find the length and width of a 52" HDTV, plug in the ratios and 52 into the Pythagorean Theorem. We know that the sides are going to be a multiple of 16 and 9, which we will call n.

$$\begin{aligned} (16n)^2+(9n)^2 &= 52^2 \\ 256n^2+81n^2 &= 2704 \\ 337n^2 &= 2704 \\ n^2 &= 8.024 \\ n &= 2.83 \end{aligned}$$



The dimensions of the TV are $16(2.83'') \times 9(2.83'')$, or $45.3'' \times 25.5''$.

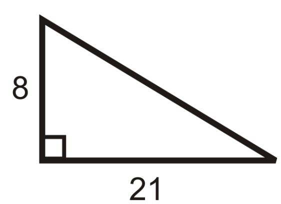
Review Questions

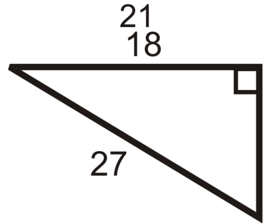
- Questions 1-9 are similar to Examples 1-3.
- Questions 10-15 are similar to Example 5 and 6.
- Questions 16-19 are similar to Example 7.
- Questions 20-25 are similar to Example 8.
- Questions 26-28 are similar to Example 9.
- Questions 29-31 are similar to Example 10.
- Questions 32 and 33 are similar to the Know What?
- Question 34 and 35 are a challenge and similar to Example 9.

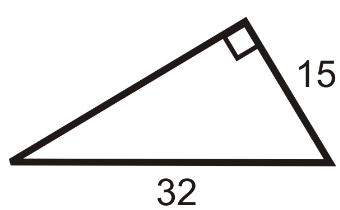
Simplify the radicals.

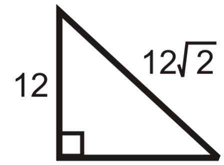
1. $2\sqrt{5} + \sqrt{20}$
2. $\sqrt{24}$
3. $\left(6\sqrt{3}\right)^2$
4. $8\sqrt{8} \cdot \sqrt{10}$
5. $\left(2\sqrt{30}\right)^2$
6. $\sqrt{320}$
7. $\frac{4\sqrt{5}}{\sqrt{6}}$
8. $\frac{12}{\sqrt{10}}$
9. $\frac{21\sqrt{5}}{9\sqrt{15}}$

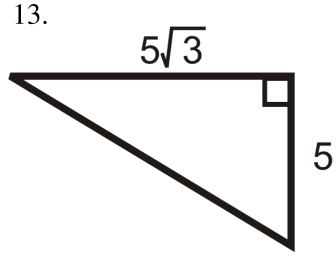
Find the length of the missing side. Simplify all radicals.

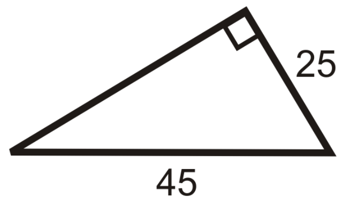
10. 

11. 

12. 

13. 

14. 

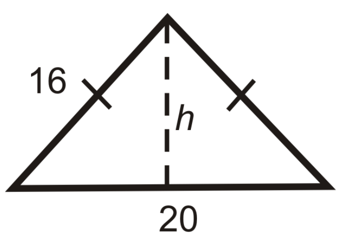
15. 

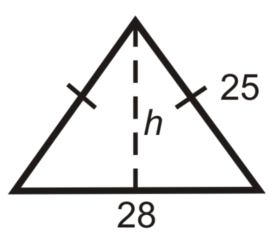
16. If the legs of a right triangle are 10 and 24, then the hypotenuse is __________.
17. If the sides of a rectangle are 12 and 15, then the diagonal is _____________.
18. If the sides of a square are 16, then the diagonal is ____________.
19. If the sides of a square are 9, then the diagonal is _____________.

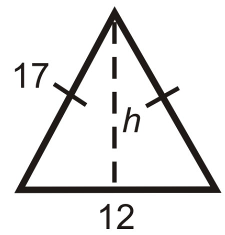
Determine if the following sets of numbers are Pythagorean Triples.

20. 12, 35, 37
21. 9, 17, 18
22. 10, 15, 21
23. 11, 60, 61
24. 15, 20, 25
25. 18, 73, 75

Find the height of each isosceles triangle below. Simplify all radicals.

26. 

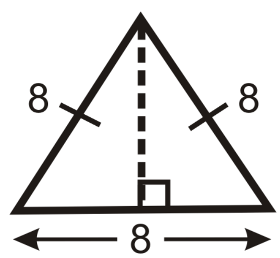
27. 



28.

Find the length between each pair of points.

29. (-1, 6) and (7, 2)
30. (10, -3) and (-12, -6)
31. (1, 3) and (-8, 16)
32. What are the length and width of a 42" HDTV? Round your answer to the nearest tenth.
33. Standard definition TVs have a length and width ratio of 4:3. What are the length and width of a 42" Standard definition TV? Round your answer to the nearest tenth.
34. ***Challenge*** An equilateral triangle is an isosceles triangle. If all the sides of an equilateral triangle are 8, find the height. Leave your answer in simplest radical form.



35. If the sides are length s, what would the height be?

Review Queue Answers

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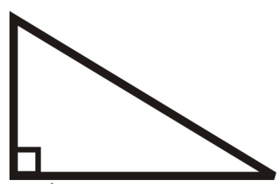
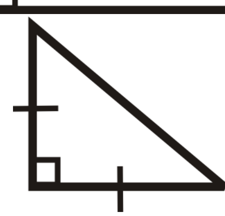


2.


3. Factors of 75: 1, 3, 5, 15, 25, 75
4. Prime Factorization of 75: $3 \cdot 5 \cdot 5$

1.3 Converse of the Pythagorean Theorem

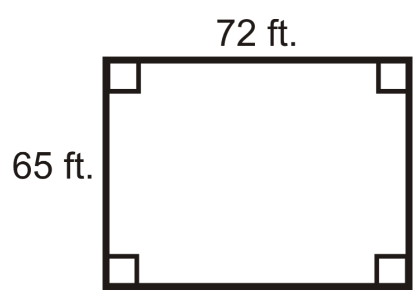
Learning Objectives

- Understand the converse of the Pythagorean Theorem.
- Identify acute and obtuse triangles from side measures.

Review Queue

a. Determine if the following sets of numbers are Pythagorean triples.
   a. 14, 48, 50
   b. 9, 40, 41
   c. 12, 43, 44
b. Do the following lengths make a right triangle? How do you know?
   a. $\sqrt{5}, 3, \sqrt{14}$
   b. $6, 2\sqrt{3}, 8$
   c. $3\sqrt{2}, 4\sqrt{2}, 5\sqrt{2}$

Know What? A friend of yours is designing a building and wantsit to be rectangular. One wall 65 ft. long and the other is 72 ft. long. How can he ensure the walls are going to be perpendicular?



Converse of the Pythagorean Theorem

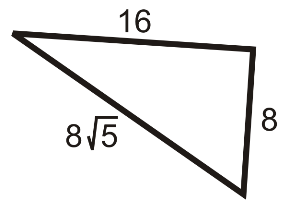
In the last lesson, you learned about the Pythagorean Theorem and how it can be used. The converse of the Pythagorean Theorem is also true. We touched on this in the last section with Example 1.

Pythagorean Theorem Converse: If the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

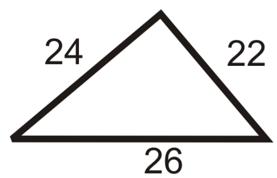
With this converse, you can use the Pythagorean Theorem to prove that a triangle is a right triangle, even if you do not know any of the triangle's angle measurements.

Example 1: Determine if the triangles below are right triangles.

a)



b)



Solution: Check to see if the three lengths satisfy the Pythagorean Theorem. Let the longest sides represent c, in the equation.

a) $a^2+b^2=c^2$
$8^2+16^2 \stackrel{?}{=} \left(8\sqrt{5}\right)^2$
$64+256 \stackrel{?}{=} 64\cdot 5$
$320=320$

The triangle is a right triangle.

b) $a^2+b^2=c^2$
$22^2+24^2 \stackrel{?}{=} 26^2$
$484+576=676$
$1060 \neq 676$

The triangle is not a right triangle.

Identifying Acute and Obtuse Triangles

We can extend the converse of the Pythagorean Theorem to determine if a triangle has an obtuse angle or is acute. We know that if the sum of the squares of the two smaller sides equals the square of the larger side, then the triangle is right. We can also interpret the outcome if the sum of the squares of the smaller sides does not equal the square of the third.

Theorem 8-3: If the sum of the squares of the two shorter sides in a right triangle is ***greater*** than the square of the longest side, then the triangle is ***acute***.

Theorem 8-4: If the sum of the squares of the two shorter sides in a right triangle is ***less*** than the square of the longest side, then the triangle is ***obtuse***.

In other words: The sides of a triangle are a, b, and c and $c > b$ and $c > a$.

If $a^2+b^2 > c^2$, then the triangle is acute.

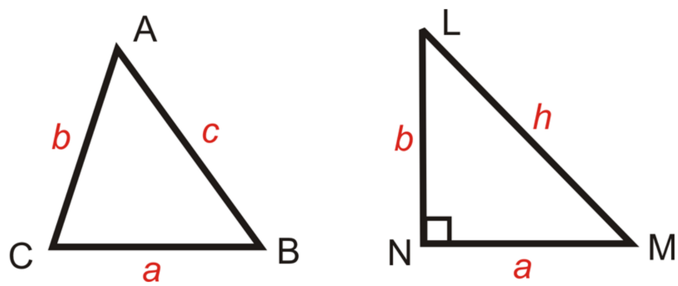
If $a^2+b^2 = c^2$, then the triangle is right.

If $a^2+b^2 < c^2$, then the triangle is obtuse.

Proof of Theorem 8-3

Given: In $\triangle ABC, a^2+b^2>c^2$, where c is the longest side.

In $\triangle LMN, \angle N$ is a right angle.



Prove: $\triangle ABC$ is an acute triangle. (all angles are less than 90°)

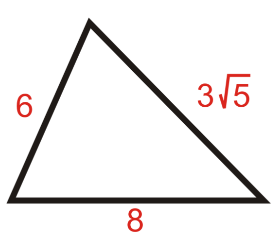
TABLE 1.1:

Statement	***Reason***
1. In $\triangle ABC, a^2+b^2>c^2$, and c is the longest side. In $\triangle LMN, \angle N$ is a right angle.	Given
2. $a^2+b^2=h^2$	Pythagorean Theorem
3. $c^2<h^2$	Transitive PoE
4. $c<h$	Take the square root of both sides
5. $\angle C$ is the largest angle in $\triangle ABC$.	The largest angle is opposite the longest side.
6. $m\angle N=90^\circ$	Definition of a right angle
7. $m\angle C<m\angle N$	SSS Inequality Theorem
8. $m\angle C<90^\circ$	Transitive PoE
9. $\angle C$ is an acute angle.	Definition of an acute angle
10. $\triangle ABC$ is an acute triangle.	If the largest angle is less than 90°, then all the angles are less than 90°.

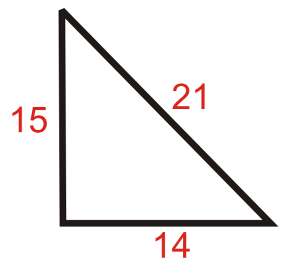
The proof of Theorem 8-4 is very similar and is in the review questions.

Example 2: Determine if the following triangles are acute, right or obtuse.

a)



b)



Solution: Set the shorter sides in each triangle equal to a and b and the longest side equal to c.

a) $6^2+(3\sqrt{5})^2 \ ? \ 8^2$
$36+45 \ ? \ 64$
$81 > 64$

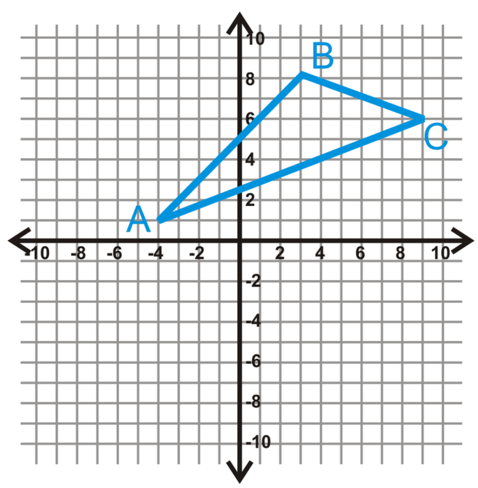
The triangle is acute.

b) $15^2+14^2 \ ? \ 21^2$
$225+196 \ ? \ 441$
$421 < 441$

The triangle is obtuse.

Example 3: Graph $A(-4,1), B(3,8)$, and $C(9,6)$. Determine if $\triangle ABC$ is acute, obtuse, or right.

Solution: This looks like an obtuse triangle, but we need proof to draw the correct conclusion. Use the distance formula to find the length of each side.



$$AB = \sqrt{(-4-3)^2+(1-8)^2} = \sqrt{49+49} = \sqrt{98} = 7\sqrt{2}$$
$$BC = \sqrt{(3-9)^2+(8-6)^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$$
$$AC = \sqrt{(-4-9)^2+(1-6)^2} = \sqrt{169+25} = \sqrt{194}$$

Now, let's plug these lengths into the Pythagorean Theorem.

$$\left(\sqrt{98}\right)^2+\left(\sqrt{40}\right)^2 \ ? \ \left(\sqrt{194}\right)^2$$
$$98+40 \ ? \ 194$$
$$138 < 194$$

$\triangle ABC$ is an obtuse triangle.

Know What? Revisited To make the walls perpendicular, find the length of the diagonal.

$$\begin{aligned} 65^2+72^2 &= c^2 \\ 4225+5184 &= c^2 \\ 9409 &= c^2 \\ 97 &= c \end{aligned}$$

In order to make the building rectangular, both diagonals must be 97 feet.

Review Questions

1. The two *shorter* sides of a triangle are 9 and 12.
   a. What would be the length of the third side to make the triangle a right triangle?
   b. What is a possible length of the third side to make the triangle acute?
   c. What is a possible length of the third side to make the triangle obtuse?
2. The two *longer* sides of a triangle are 24 and 25.
   a. What would be the length of the third side to make the triangle a right triangle?
   b. What is a possible length of the third side to make the triangle acute?
   c. What is a possible length of the third side to make the triangle obtuse?
3. The lengths of the sides of a triangle are $8x, 15x$, and $17x$. Determine if the triangle is acute, right, or obtuse.

Determine if the following lengths make a right triangle.

4. 15, 20, 25
5. 20, 25, 30
6. $8\sqrt{3}, 6, 2\sqrt{39}$

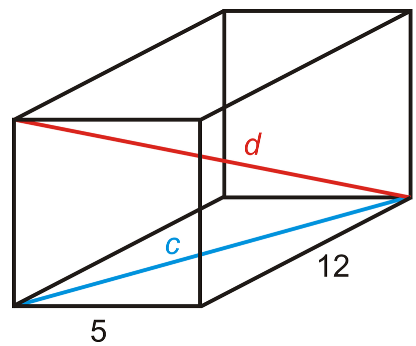
Determine if the following triangles are acute, right or obtuse.

7. 7, 8, 9
8. 14, 48, 50
9. 5, 12, 15
10. 13, 84, 85
11. 20, 20, 24
12. 35, 40, 51
13. 39, 80, 89
14. 20, 21, 38
15. 48, 55, 76

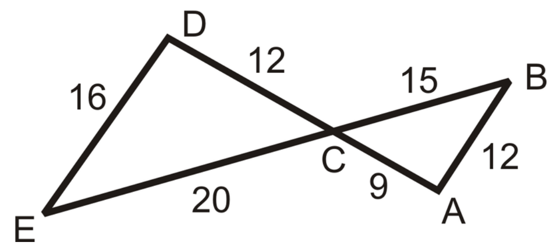
Graph each set of points and determine if $\triangle ABC$ is acute, right, or obtuse.

16. $A(3,-5), B(-5,-8), C(-2,7)$
17. $A(5,3), B(2,-7), C(-1,5)$
18. ***Writing*** Explain the two different ways you can show that a triangle in the coordinate plane is a right triangle.

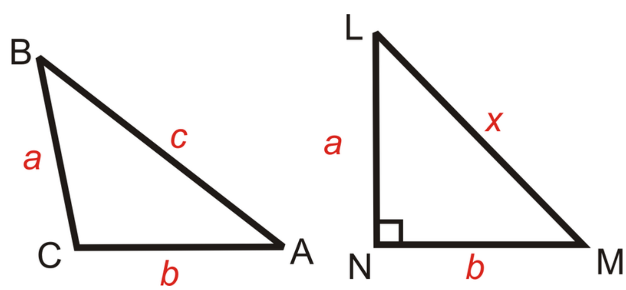
The figure to the right is a rectangular prism. All sides (or faces) are either squares (the front and back) or rectangles (the four around the middle). All sides are perpendicular.



19. Find c.
20. Find d.



21. ***Writing*** Explain why $m\angle A = 90^\circ$.
22. Fill in the blanks for the proof of Theorem 8-4.



Given: In $\triangle ABC, a^2+b^2<c^2$, where c is the longest side. In $\triangle LMN, \angle N$ is a right angle. Prove: $\triangle ABC$ is an obtuse triangle. (one angle is greater than 90°)

TABLE 1.2:

Statement	***Reason***
1. In $\triangle ABC, a^2+b^2<c^2$, and c is the longest side. In $\triangle LMN, \angle N$ is a right angle.	
2. $a^2+b^2=h^2$	
3. $c^2>h^2$	
4.	
5. $\angle C$ is the largest angle in $\triangle ABC$.	
6. $m\angle N = 90^\circ$	
7. $m\angle C > m\angle N$	
8.	Transitive PoE
9. $\angle C$ is an obtuse angle.	
10. $\triangle ABC$ is an obtuse triangle.	

Given $\overline{AB}$, with $A(3,3)$ and $B(2,-3)$ determine whether the given point C in problems 23-25 makes an acute, right or obtuse triangle.

23. $C(3,-3)$
24. $C(4,-1)$
25. $C(5,-2)$

Given $\overline{AB}$, with $A(-2,5)$ and $B(1,-3)$ find at least two possible points, C, such that $\triangle ABC$ is

26. right, with right $\angle C$.
27. acute, with acute $\angle C$.
28. obtuse, with obtuse $\angle C$.
29. ***Construction***
    a. Draw $\overline{AB}$, such that $AB = 3\ in$.
    b. Draw $\overrightarrow{AD}$ such that $\angle BAD < 90^\circ$.
    c. Construct a line through B which is perpendicular to $\overrightarrow{AD}$, label the intersection C.
    d. $\triangle ABC$ is a right triangle with right $\angle C$.
30. Is the triangle you made unique? In other words, could you have multiple different outcomes with the same AB? Why or why not? You may wish to experiment to find out.
31. Why do the instructions specifically require that $\angle BAD < 90^\circ$?
32. Describe how this construction could be changed so that $\angle B$ is the right angle in the triangle.

Review Queue Answers

a. Yes
b. Yes
c. No

a. Yes
b. No
c. Yes

1.4 Using Similar Right Triangles

Learning Objectives

- Identify similar triangles inscribed in a larger triangle.
- Evaluate the geometric mean.
- Find the length of an altitude or leg using the geometric mean.

Review Queue

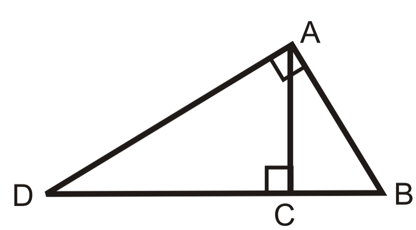
a. If two triangles are right triangles, does that mean they are similar? Explain.
b. If two triangles are isosceles right triangles, does that mean they are similar? Explain.
c. Solve the ratio: $\frac{3}{x} = \frac{x}{27}$.
d. If the legs of an isosceles right triangle are 4, find the length of the hypotenuse. Draw a picture and simplify the radical.

Know What? In California, the average home price increased 21.3% in 2004 and another 16.0% in 2005. What is the average rate of increase for these two years?

Inscribed Similar Triangles

You may recall that if two objects are similar, corresponding angles are congruent and their sides are proportional in length. Let's look at a right triangle, with an altitude drawn from the right angle.

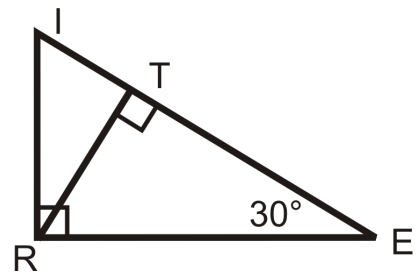
There are three right triangles in this picture, $\triangle ADB, \triangle CDA$, and $\triangle CAB$. Both of the two smaller triangles are similar to the larger triangle because they each share an angle with $\triangle ADB$. That means all three triangles are similar to each other.



Theorem 8-5: If an altitude is drawn from the right angle of any right triangle, then the two triangles formed are similar to the original triangle and all three triangles are similar to each other.

The proof of Theorem 8-5 is in the review questions.

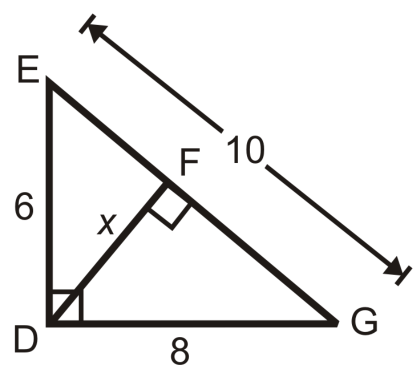
Example 1: Write the similarity statement for the triangles below.



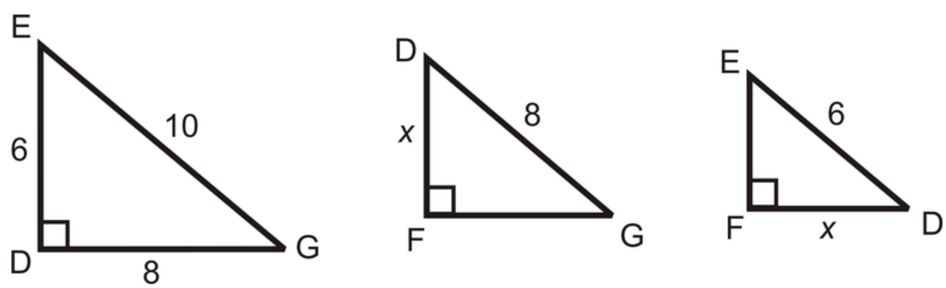
Solution: If $m\angle E = 30^\circ$, then $m\angle I = 60^\circ$ and $m\angle TRE = 60^\circ$. $m\angle IRT = 30^\circ$ because it is complementary to $\angle TRE$. Line up the congruent angles in the similarity statement. $\triangle IRE \sim \triangle ITR \sim \triangle RTE$

We can also use the side proportions to find the length of the altitude.

Example 2: Find the value of x.



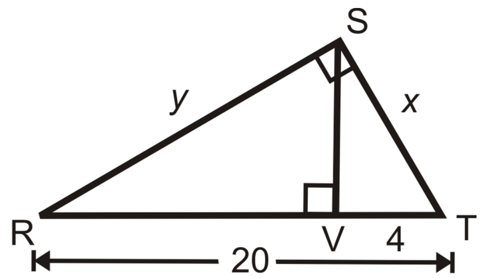
Solution: First, let's separate the triangles to find the corresponding sides.



Now we can set up a proportion.

$$\frac{\text{shorter leg in } \triangle EDG}{\text{shorter leg in } \triangle DFG} = \frac{\text{hypotenuse in } \triangle EDG}{\text{hypotenuse in } \triangle DFG}$$
$$\frac{6}{x} = \frac{10}{8}$$
$$48 = 10x$$
$$4.8 = x$$

Example 3: Find the value of x.



Solution: Let's set up a proportion.

$$\begin{aligned}
\frac{\text{shorter leg in } \triangle SVT}{\text{shorter leg in } \triangle RST} &= \frac{\text{hypotenuse in } \triangle SVT}{\text{hypotenuse in } \triangle RST} \\
\frac{4}{x} &= \frac{x}{20} \\
x^2 &= 80 \\
x &= \sqrt{80} = 4\sqrt{5}
\end{aligned}$$

Example 4: Find the value of y in $\triangle RST$ above.

Solution: Use the Pythagorean Theorem.

$$\begin{aligned}
y^2 + \left(4\sqrt{5}\right)^2 &= 20^2 \\
y^2 + 80 &= 400 \\
y^2 &= 320 \\
y &= \sqrt{320} = 8\sqrt{5}
\end{aligned}$$

The Geometric Mean

You are probably familiar with the arithmetic mean, which ***divides the sum*** of n numbers by n. This is commonly used to determine the average test score for a group of students.

The geometric mean is a different sort of average, which takes the n^{th} ***root of the product*** of n numbers. In this text, we will primarily compare two numbers, so we would be taking the square root of the product of two numbers. This mean is commonly used with rates of increase or decrease.

Geometric Mean: The geometric mean of two positive numbers a and b is the number x, such that $\frac{a}{x} = \frac{x}{b}$ or $x^2 = ab$ and $x = \sqrt{ab}$.

Example 5: Find the geometric mean of 24 and 36.

Solution: $x = \sqrt{24 \cdot 36} = \sqrt{12 \cdot 2 \cdot 12 \cdot 3} = 12\sqrt{6}$

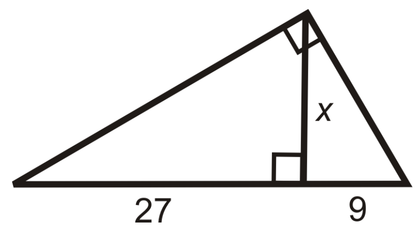
Example 6: Find the geometric mean of 18 and 54.

Solution: $x = \sqrt{18 \cdot 54} = \sqrt{18 \cdot 18 \cdot 3} = 18\sqrt{3}$

Notice that in both of these examples, we did not actually multiply the two numbers together, but kept them separate. This made it easier to simplify the radical.

A practical application of the geometric mean is to find the altitude of a right triangle.

Example 7: Find the value of x.



Solution: Using similar triangles, we have the proportion

$$\frac{\text{shortest leg of smallest } \triangle}{\text{shortest leg of middle } \triangle} = \frac{\text{longer leg of smallest } \triangle}{\text{longer leg of middle } \triangle}$$

$$\frac{9}{x} = \frac{x}{27}$$

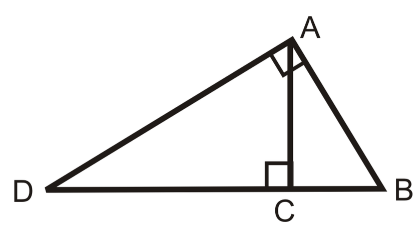
$$x^2 = 243$$

$$x = \sqrt{243} = 9\sqrt{3}$$

In Example 7, $\frac{9}{x} = \frac{x}{27}$ is in the definition of the geometric mean. So, the altitude is the geometric mean of the two segments that it divides the hypotenuse into.

Theorem 8-6: In a right triangle, the altitude drawn from the right angle to the hypotenuse divides the hypotenuse into two segments. The length of the altitude is the geometric mean of these two segments.

Theorem 8-7: In a right triangle, the altitude drawn from the right angle to the hypotenuse divides the hypotenuse into two segments. The length of each leg of the right triangle is the geometric mean of the lengths of the hypotenuse and the segment of the hypotenuse that is adjacent to the leg.



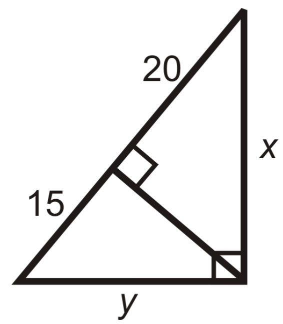
In other words

Theorem 8-6: $\frac{BC}{AC} = \frac{AC}{DC}$

Theorem 8-7: $\frac{BC}{AB} = \frac{AB}{DB}$ and $\frac{DC}{AD} = \frac{AD}{DB}$

Both of these theorems are proved using similar triangles.

Example 8: Find the value of x and y.



Solution: Use theorem 8-7 to solve for x and y.

$$\begin{aligned} \frac{20}{x} &= \frac{x}{35} \\ x^2 &= 20 \cdot 35 \\ x &= \sqrt{20 \cdot 35} \\ x &= 10\sqrt{7} \end{aligned} \qquad \begin{aligned} \frac{15}{y} &= \frac{y}{35} \\ y^2 &= 15 \cdot 35 \\ y &= \sqrt{15 \cdot 35} \\ y &= 5\sqrt{21} \end{aligned}$$

You could also use the Pythagorean Theorem to solve for y, once x has been solved for.

$$\begin{aligned} \left(10\sqrt{7}\right)^2 + y^2 &= 35^2 \\ 700 + y^2 &= 1225 \\ y &= \sqrt{525} = 5\sqrt{21} \end{aligned}$$

Either method is acceptable.

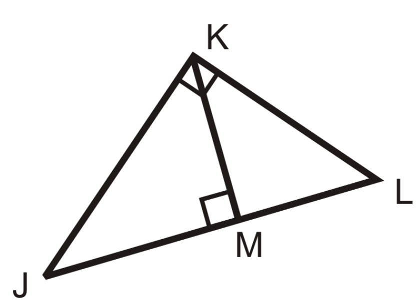
Know What? Revisited The average rate of increase can be found by using the geometric mean.

$$x = \sqrt{0.213 \cdot 0.16} = 0.1846$$

Over the two year period, housing prices increased 18.46%.

Review Questions

Use the diagram to answer questions 1-4.

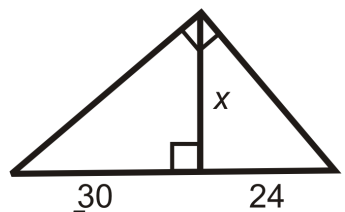


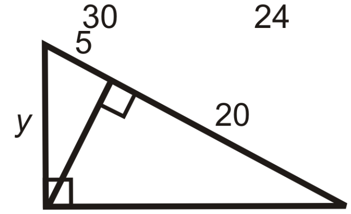
1. Write the similarity statement for the three triangles in the diagram.
2. If $JM = 12$ and $ML = 9$, find KM.
3. Find JK.
4. Find KL.

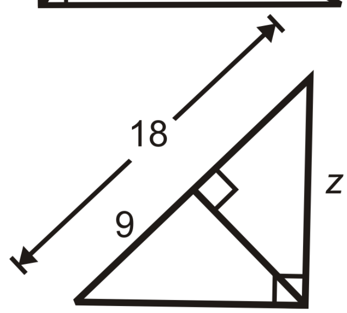
Find the geometric mean between the following two numbers. Simplify all radicals.

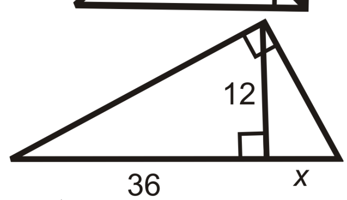
5. 16 and 32
6. 45 and 35
7. 10 and 14
8. 28 and 42
9. 40 and 100
10. 51 and 8

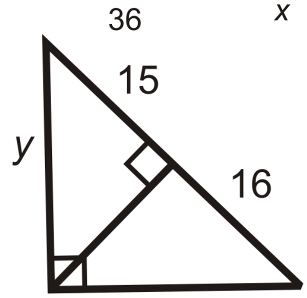
Find the length of the missing variable(s). Simplify all radicals.

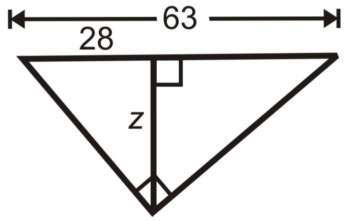
11. 

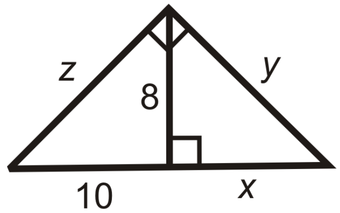
12. 

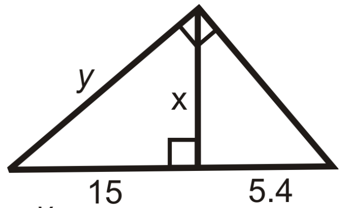
13. 

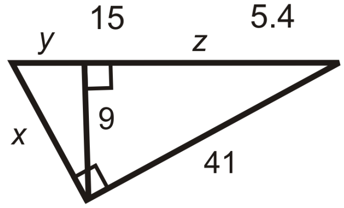
14. 

15. 

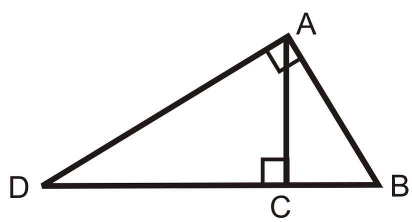
16. 

17. 

18. 

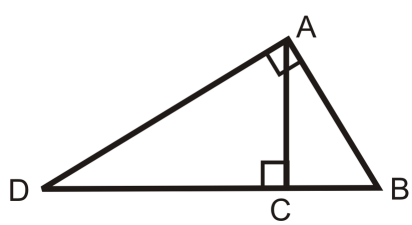
19. 

20. Write a proof for Theorem 8-5.



Given: $\triangle ABD$ with $\overline{AC} \perp \overline{DB}$ and $\angle DAB$ is a right angle. Prove: $\triangle ABD \sim \triangle CBA \sim \triangle CAD$

21. Fill in the blanks for the proof of Theorem 8-7.



Given: $\triangle ABD$ with $\overline{AC} \perp \overline{DB}$ and $\angle DAB$ is a right angle. Prove: $\frac{BC}{AB} = \frac{AB}{DB}$

TABLE 1.3:

Statement	***Reason***
1. $\triangle ABD$ with $\overline{AC} \perp \overline{DB}$ and $\angle DAB$ is a right angle.	
2. $\triangle ABD \sim \triangle CBA \sim \triangle CAD$	
3. $\frac{BC}{AB} = \frac{AB}{DB}$	

22. Last year Poorva's rent increased by 5% and this year her landlord wanted to raise her rent by 7.5%. What is the average rate at which her landlord has raised her rent over the course of these two years?

23. Mrs. Smith teaches AP Calculus. Between the first and second years she taught the course her students' average score improved by 12%. Between the second and third years, the scores increased by 9%. What is the average rate of improvement in her students' scores?
24. According to the US Census Bureau, http://www.census.gov/ipc/www/idb/country.php the rate of growth of the US population was 0.8% and in 2009 it was 1.0%. What was the average rate of population growth during that time period?

Algebra Connection A geometric sequence is a sequence of numbers in which each successive term is determined by multiplying the previous term by the common ratio. An example is the sequence 1, 3, 9, 27, ... Here each term is multiplied by 3 to get the next term in the sequence. Another way to look at this sequence is to compare the ratios of the consecutive terms.

25. Find the ratio of the 2^{nd} to 1^{st} terms and the ratio of the 3^{rd} to 2^{nd} terms. What do you notice? Is this true for the next set (4^{th} to 3^{rd} terms)?
26. Given the sequence 4, 8, 16,..., if we equate the ratios of the consecutive terms we get: $\frac{8}{4} = \frac{16}{8}$. This means that 8 is the __________________ of 4 and 16. We can generalize this to say that every term in a geometric sequence is the __________________ of the previous and subsequent terms.

Use what you discovered in problem 26 to find the middle term in the following geometric sequences.

27. 5, _____, 20
28. 4, _____, 100
29. 2, _____, $\frac{1}{2}$
30. We can use what we have learned in this section in another proof of the Pythagorean Theorem. Use the diagram to fill in the blanks in the proof below.

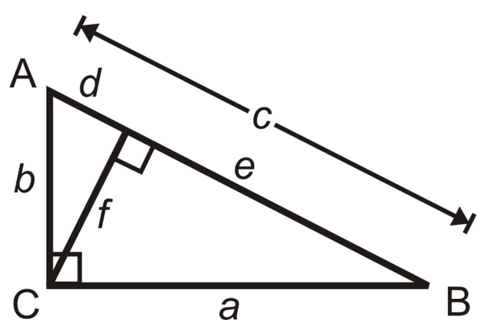


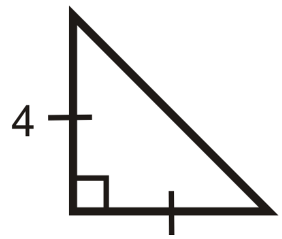
TABLE 1.4:

Statement	***Reason***
1. $\frac{e}{a} = \frac{?}{d+e}$ and $\frac{d}{b} = \frac{b}{?}$	Theorem 8-7
2. $a^2 = e(d+e)$ and $b^2 = d(d+e)$	?
3. $a^2 + b^2 = ?$	Combine equations from #2.
4. ?	Distributive Property
5. $c = d+e$	?
6. ?	Substitution PoE

Review Queue Answers

a. No, another angle besides the right angles must also be congruent.

b. Yes, the three angles in an isosceles right triangle are $45^\circ, 45^\circ$, and 90°. Isosceles right triangles will always be similar.
c. $\frac{3}{x} = \frac{x}{27} \rightarrow x^2 = 81 \rightarrow x = \pm 9$
d. $4^2 + 4^2 = h^2$
$h = \sqrt{32} = 4\sqrt{2}$



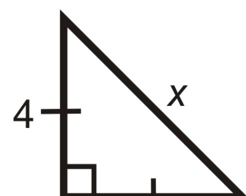
1.5 Special Right Triangles

Learning Objectives

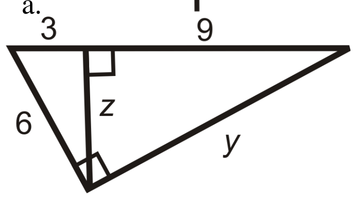
- Identify and use the ratios involved with isosceles right triangles.
- Identify and use the ratios involved with 30-60-90 triangles.

Review Queue

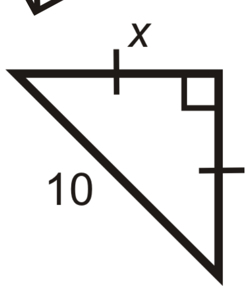
Find the value of the missing variable(s). Simplify all radicals.



a.



b.



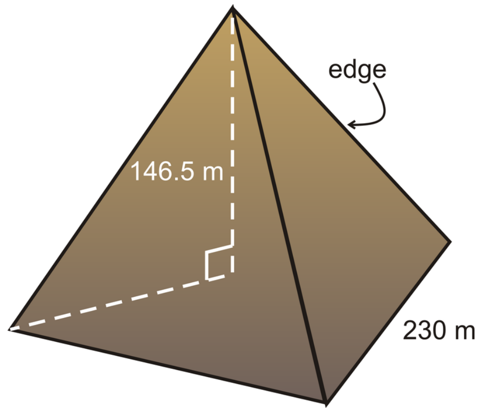
c.

d. Do the lengths 6, 6, and $6\sqrt{2}$ make a right triangle?
e. Do the lengths $3, 3\sqrt{3}$, and 6 make a right triangle?

Know What? The Great Giza Pyramid is a pyramid with a square base and four isosceles triangles that meet at a point. It is thought that the original height was 146.5 meters and the base edges were 230 meters.

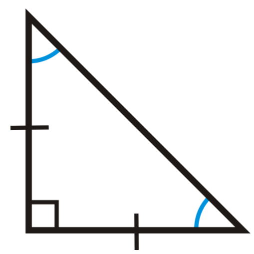
First, find the length of the edge of the isosceles triangles. Then, determine if the isosceles triangles are also equilateral triangles. Round your answers to the nearest tenth.

You can assume that the height of the pyramid is from the center of the square base and is a vertical line.



Isosceles Right Triangles

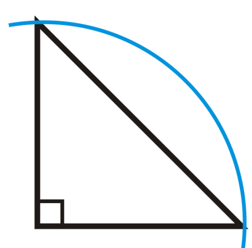
There are two types of special right triangles, based on their angle measures. The first is an isosceles right triangle. Here, the legs are congruent and, by the Base Angles Theorem, the base angles will also be congruent. Therefore, the angle measures will be $90^\circ, 45^\circ$, and 45°. You will also hear an isosceles right triangle called a 45-45-90 triangle. Because the three angles are always the same, all isosceles right triangles are similar.



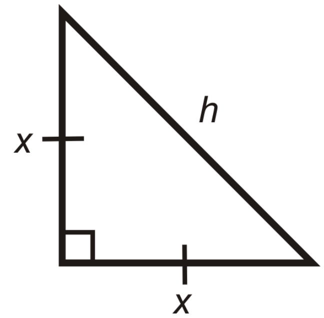
Investigation 8-2: Properties of an Isosceles Right Triangle

Tools Needed: Pencil, paper, compass, ruler, protractor

a. Construct an isosceles right triangle with 2 in legs. Use the SAS construction that you learned in Chapter 4.



b. Find the measure of the hypotenuse. What is it? Simplify the radical.
c. Now, let's say the legs are of length x and the hypotenuse is h. Use the Pythagorean Theorem to find the hypotenuse. What is it? How is this similar to your answer in #2?



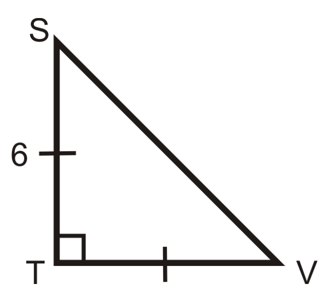
$$x^2 + x^2 = h^2$$
$$2x^2 = h^2$$
$$x\sqrt{2} = h$$

45-45-90 Corollary: If a triangle is an isosceles right triangle, then its sides are in the extended ratio $x : x : x\sqrt{2}$.

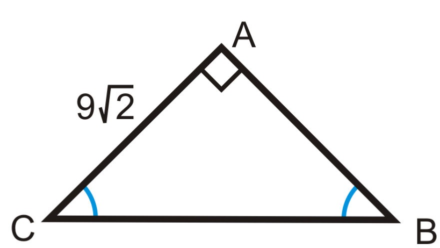
Step 3 in the above investigation proves the 45-45-90 Triangle Theorem. So, anytime you have a right triangle with congruent legs or congruent angles, then the sides will always be in the ratio $x : x : x\sqrt{2}$. The hypotenuse is always $x\sqrt{2}$ because that is the longest length. This is a specific case of the Pythagorean Theorem, so it will still work, if for some reason you forget this corollary.

Example 1: Find the length of the missing sides.

a)



b)



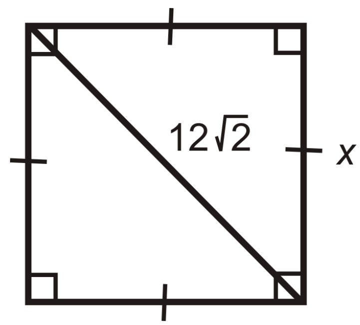
Solution: Use the $x : x : x\sqrt{2}$ ratio.

a) $TV = 6$ because it is equal to ST. So, $SV = 6\sqrt{2}$.

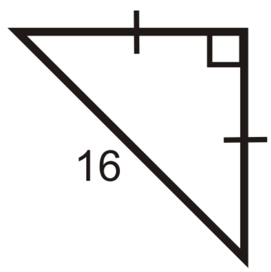
b) $AB = 9\sqrt{2}$ because it is equal to AC. So, $BC = 9\sqrt{2} \cdot \sqrt{2} = 9 \cdot 2 = 18$.

Example 2: Find the length of x.

a)



b)



Solution: Again, use the $x:x:x\sqrt{2}$ ratio, but in these two we are given the hypotenuse. We need to solve for x in the ratio.

a) $12\sqrt{2} = x\sqrt{2}$
$12 = x$

b) $x\sqrt{2} = 16$
$x = \frac{16}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{16\sqrt{2}}{2} = 8\sqrt{2}$

In part b, we ***rationalized the denominator***. Whenever there is a radical in the denominator of a fraction, multiply the top and bottom by that radical. This will cancel out the radical from the denominator and reduce the fraction.

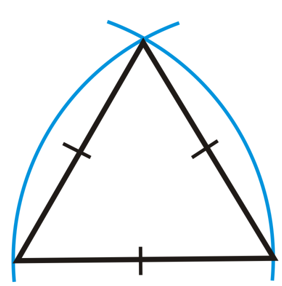
30-60-90 Triangles

The second special right triangle is called a 30-60-90 triangle, after the three angles. To construct a 30-60-90 triangle, start with an equilateral triangle.

Investigation 8-3: Properties of a 30-60-90 Triangle

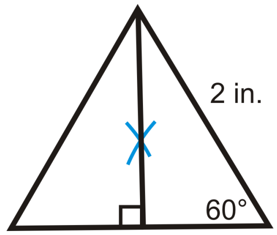
Tools Needed: Pencil, paper, ruler, compass

1. Construct an equilateral triangle with 2 in sides.



2. Draw or construct the altitude from the top vertex to the base for two congruent triangles.

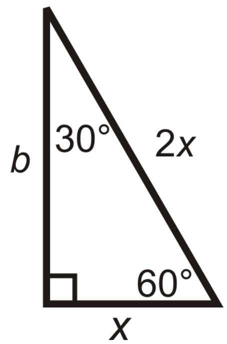
3. Find the measure of the two angles at the top vertex and the length of the shorter leg.



The top angles are each 30° *and the shorter leg is 1 in because the altitude of an equilateral triangle is also the angle and perpendicular bisector.*

4. Find the length of the longer leg, using the Pythagorean Theorem. Simplify the radical.

5. Now, let's say the shorter leg is length x and the hypotenuse is $2x$. Use the Pythagorean Theorem to find the longer leg. What is it? How is this similar to your answer in #4?



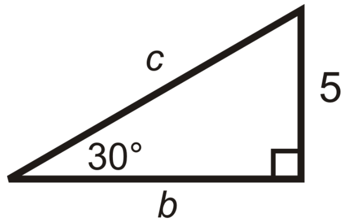
$$x^2 + b^2 = (2x)^2$$
$$x^2 + b^2 = 4x^2$$
$$b^2 = 3x^2$$
$$b = x\sqrt{3}$$

30-60-90 Corollary: If a triangle is a 30-60-90 triangle, then its sides are in the extended ratio $x : x\sqrt{3} : 2x$.

Step 5 in the above investigation proves the 30-60-90 Corollary. The shortest leg is always x, the longest leg is always $x\sqrt{3}$, and the hypotenuse is always $2x$. If you ever forget this corollary, then you can still use the Pythagorean Theorem.

Example 3: Find the length of the missing sides.

a)



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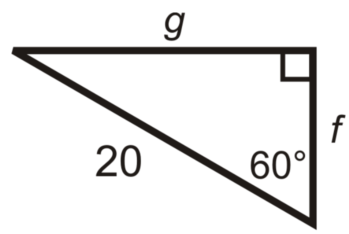


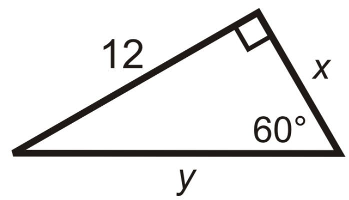
Solution: In part a, we are given the shortest leg and in part b, we are given the hypotenuse.

a) If $x = 5$, then the longer leg, $b = 5\sqrt{3}$, and the hypotenuse, $c = 2(5) = 10$.

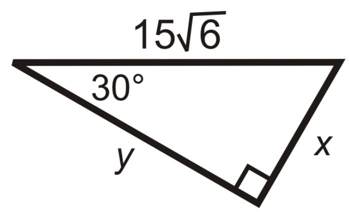
b) Now, $2x = 20$, so the shorter leg, $f = 10$, and the longer leg, $g = 10\sqrt{3}$.

Example 4: Find the value of x and y.

a)



b)

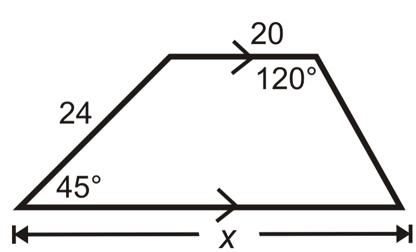


Solution: In part a, we are given the longer leg and in part b, we are given the hypotenuse.

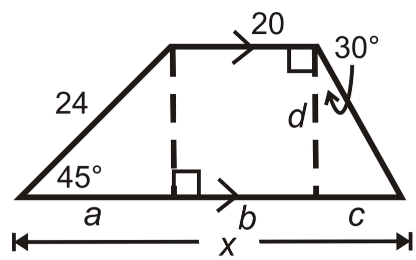
a) $x\sqrt{3} = 12$
$x = \frac{12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$
Then, the hypotenuse would be
$y = 2\left(4\sqrt{3}\right) = 8\sqrt{3}$

b) $2x = 15\sqrt{6}$
$x = \frac{15\sqrt{6}}{2}$
The, the longer leg would be
$y = \left(\frac{15\sqrt{6}}{2}\right) \cdot \sqrt{3} = \frac{15\sqrt{18}}{2} = \frac{45\sqrt{2}}{2}$

Example 5: Find the measure of x.



Solution: Think of this trapezoid as a rectangle, between a 45-45-90 triangle and a 30-60-90 triangle.



From this picture, $x = a+b+c$. First, find a, which is a leg of an isosceles right triangle.

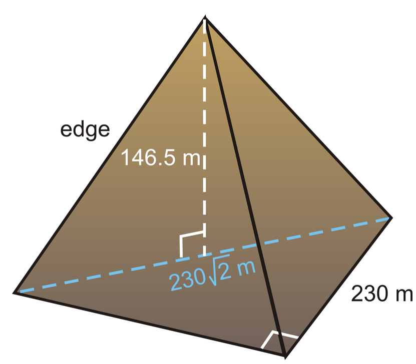
$$a = \frac{24}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{24\sqrt{2}}{2} = 12\sqrt{2}$$

$a = d$, so we can use this to find c, which is the shorter leg of a 30-60-90 triangle.

$$c = \frac{12\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{6}}{3} = 4\sqrt{6}$$

$b = 20$, so $x = 12\sqrt{2}+20+4\sqrt{6}$. Nothing simplifies, so this is how we leave our answer.

Know What? Revisited The line that the vertical height is perpendicular to is the diagonal of the square base. This length (blue) is the same as the hypotenuse of an isosceles right triangle because half of a square is an isosceles right triangle. So, the diagonal is $230\sqrt{2}$. Therefore, the base of the right triangle with 146.5 as the leg is half of $230\sqrt{2}$ or $115\sqrt{2}$. Do the Pythagorean Theorem to find the edge.



$$edge = \sqrt{\left(115\sqrt{2}\right)^2 + 146.5^2} \approx 218.9\ m$$

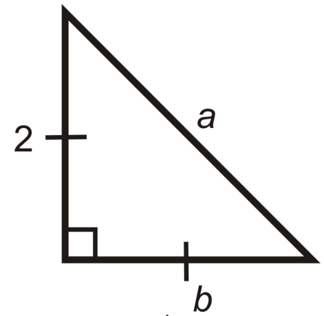
In order for the sides to be equilateral triangles, this length should be 230 meters. It is not, so the triangles are isosceles.

Review Questions

1. In an isosceles right triangle, if a leg is x, then the hypotenuse is __________.

2. In a 30-60-90 triangle, if the shorter leg is x, then the longer leg is __________ and the hypotenuse is _____-______.
3. A square has sides of length 15. What is the length of the diagonal?
4. A square's diagonal is 22. What is the length of each side?
5. A rectangle has sides of length 4 and $4\sqrt{3}$. What is the length of the diagonal?
6. A baseball diamond is a square with 90 foot sides. What is the distance from home base to second base? (HINT: It's the length of the diagonal).

For questions 7-18, find the lengths of the missing sides.



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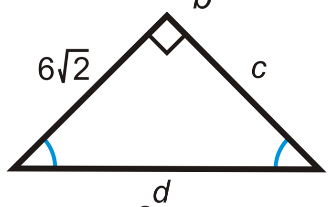

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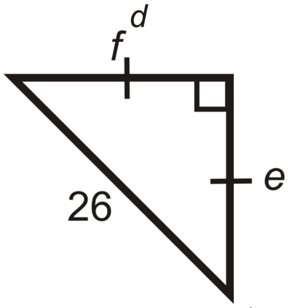

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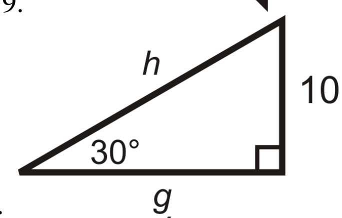

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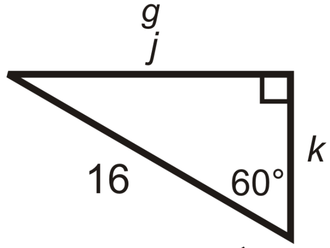

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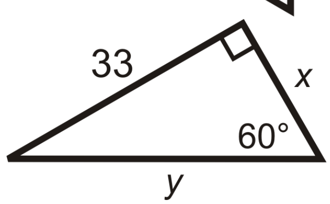

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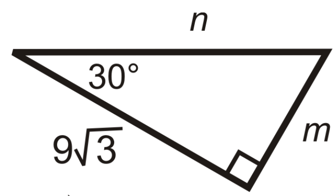
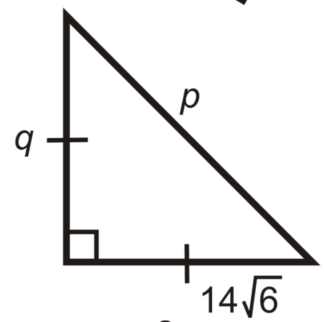
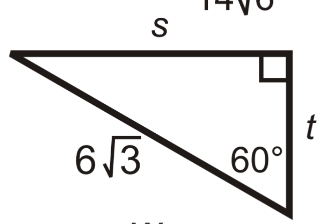
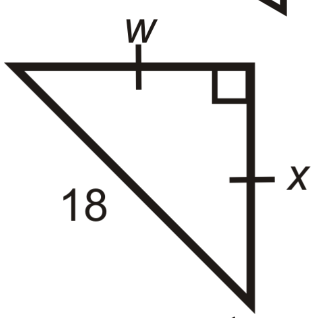
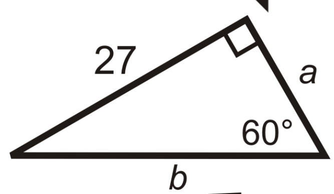
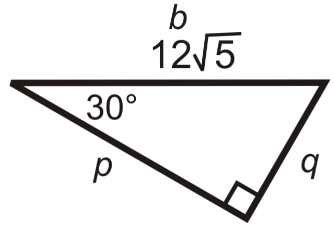
13. 

14. 

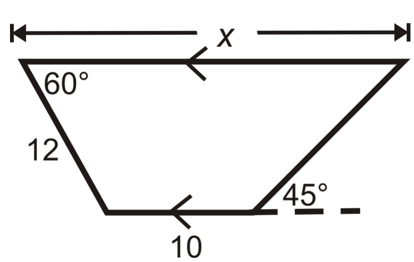
15. 

16. 

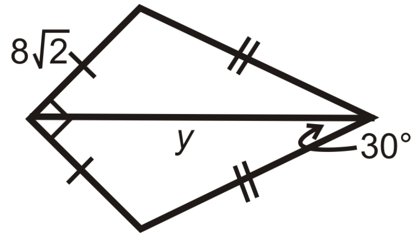
17. 

18. 

19. Do the lengths $8\sqrt{2}, 8\sqrt{6}$, *and* $16\sqrt{2}$ make a special right triangle? If so, which one?
20. Do the lengths $4\sqrt{3}, 4\sqrt{6}$ *and* $8\sqrt{3}$ make a special right triangle? If so, which one?
21. Find the measure of x.



22. Find the measure of y.



23. What is the ratio of the sides of a rectangle if the diagonal divides the rectangle into two 30-60-90 triangles?
24. What is the length of the sides of a square with diagonal 8 in?

For questions 25-28, it might be helpful to recall #25 from section 8.1.

25. What is the height of an equilateral triangle with sides of length 3 in?
26. What is the area of an equilateral triangle with sides of length 5 ft?
27. A regular hexagon has sides of length 3 in. What is the area of the hexagon? (*Hint: the hexagon is made up a 6 equilateral triangles.*)
28. The area of an equilateral triangle is $36\sqrt{3}$. What is the length of a side?
29. If a road has a grade of 30°, this means that its angle of elevation is 30°. If you travel 1.5 miles on this road, how much elevation have you gained in feet (5280 ft = 1 mile)?
30. Four isosceles triangles are formed when both diagonals are drawn in a square. If the length of each side in the square is s, what are the lengths of the legs of the isosceles triangles?

Review Queue Answers

a. $4^2+4^2=x^2$
$32=x^2$
$x=4\sqrt{2}$

b. $3^2+z^2=6^2$ $\quad\left(3\sqrt{3}\right)^2+9^2=y^2$
$z^2=27$ $\quad 108=y^2$
$z=3\sqrt{3}$ $\quad y=6\sqrt{3}$

c. $x^2+x^2=10^2$
$2x^2=100$
$x^2=50$
$x=5\sqrt{2}$

d. Yes, $6^2+6^2=\left(6\sqrt{2}\right)^2 \rightarrow 36+36=72$

e. Yes, $3^2+\left(3\sqrt{3}\right)^2=6^2 \rightarrow 9+27=36$

1.6 The Distance Formula

Learning Objectives

- Find the distance between two points.
- Find the shortest distance between a point and a line and two parallel lines.
- Determine the equation of a perpendicular bisector of a line segment in the coordinate plane.

Review Queue

1. What is the equation of the line between (-1, 3) and (2, -9)?
2. Find the equation of the line that is perpendicular to $y = -2x + 5$ through the point (-4, -5).
3. Find the equation of the line that is parallel to $y = \frac{2}{3}x - 7$ through the point (3, 8).

Know What? The shortest distance between two points is a straight line. To the right is an example of how far apart cities are in the greater Los Angeles area. There are always several ways to get somewhere in Los Angeles. Here, we have the distances between Los Angeles and Orange. Which distance is the shortest? Which is the longest?

The Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) can be defined as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This formula will be derived in Chapter 9.

Example 1: Find the distance between (4, -2) and (-10, 3).

Solution: Plug in (4, -2) for (x_1, y_1) and (-10, 3) for (x_2, y_2) and simplify.

$$\begin{aligned} d &= \sqrt{(-10-4)^2+(3+2)^2} \\ &= \sqrt{(-14)^2+(5^2)} \qquad \textit{Distances are always positive!} \\ &= \sqrt{196+25} \\ &= \sqrt{221} \approx 14.87 \textit{ units} \end{aligned}$$

Example 2: The distance between two points is 4 units. One point is (1, -6). What is the second point? You may assume that the second point is made up of integers.

Solution: We will still use the distance formula for this problem, however, we know d and need to solve for $(x_2,\ y_2)$.

$$\begin{aligned} 4 &= \sqrt{(1-x_2)^2+(-6-y_2)^2} \\ 16 &= (1-x_2)^2+(-6-y_2)^2 \end{aligned}$$

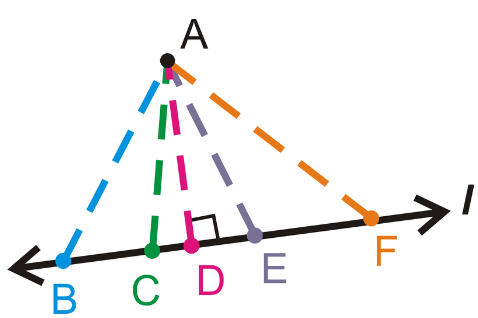
At this point, we need to figure out two square numbers that add up to 16. The only two square numbers that add up to 16 are $16+0$.

$$\begin{array}{lllllll} 16 = \underbrace{(1-x_2)^2}_{4^2} + \underbrace{(-6-y_2)^2}_{0^2} & & & \text{or} & 16 = \underbrace{(1-x_2)^2}_{0^2} + \underbrace{(-6-y_2)^2}_{4^2} & & \\ 1-x_2 = \pm 4 & & -6-y_2 = 0 & & 1-x_2 = 0 & & -6-y_2 = \pm 4 \\ -x_2 = -5 \text{ or } 3 & \text{and} & -y_2 = 6 & \text{or} & -x_2 = -1 & \text{and} & y_2 = 10 \text{ or } 2 \\ x_2 = 5 \text{ or } -3 & & y_2 = -6 & & x_2 = 1 & & y_2 = -10 \text{ or } -2 \end{array}$$

Therefore, the second point could have 4 possibilities: (5, -6), (-3, -6), (1, -10), and (1, -2).

Shortest Distance between a Point and a Line

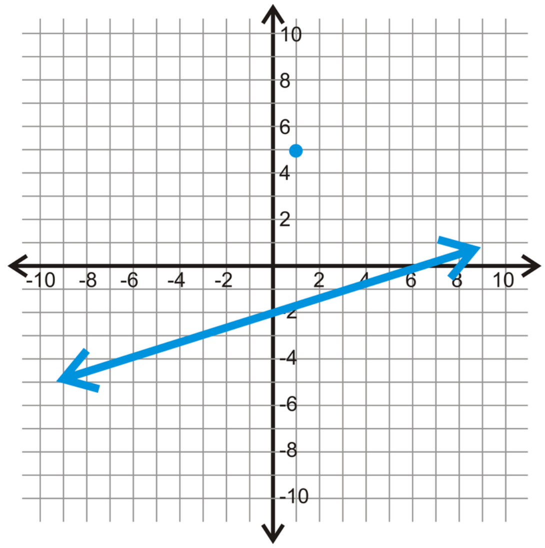
We know that the shortest distance between two points is a straight line. This distance can be calculated by using the distance formula. Let's extend this concept to the shortest distance between a point and a line.



Just by looking at a few line segments from A to line l, we can tell that the shortest distance between a point and a line is the ***perpendicular line*** between them. Therefore, AD is the shortest distance between A and line l.

Putting this onto a graph can be a little tougher.

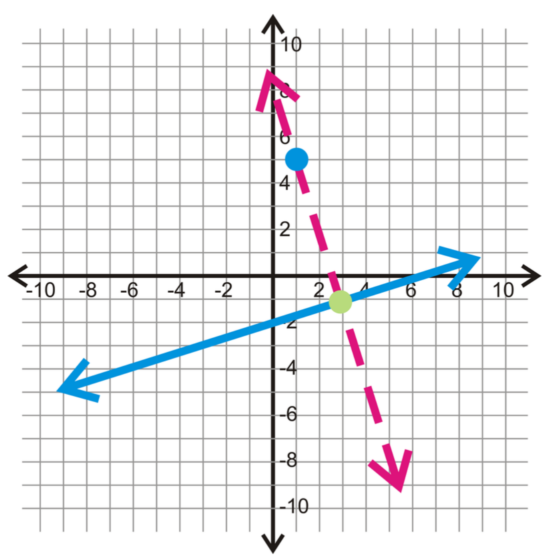
Example 3: Determine the shortest distance between the point (1, 5) and the line $y = \frac{1}{3}x - 2$.



Solution: First, graph the line and point. Second determine the equation of the perpendicular line. The opposite sign and reciprocal of $\frac{1}{3}$ is -3, so that is the slope. We know the line must go through the given point, (1, 5), so use that to find the $y-$intercept.

$$\begin{aligned} y &= -3x+b \\ 5 &= -3(1)+b \qquad \text{The equation of the line is } y=-3x+8. \\ 8 &= b \end{aligned}$$

Next, we need to find the point of intersection of these two lines. By graphing them on the same axes, we can see that the point of intersection is (3, -1), the green point.

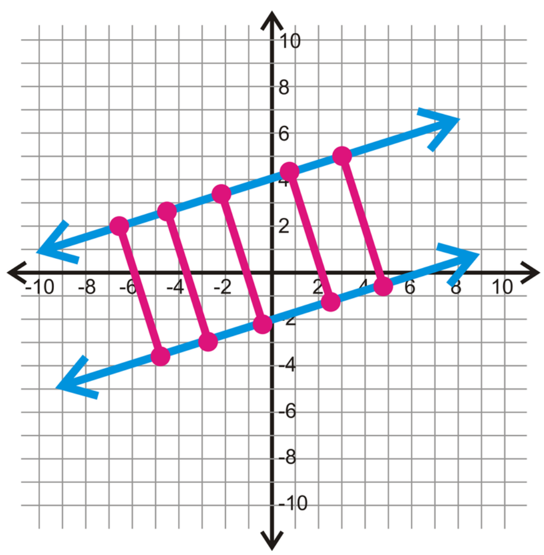


Finally, plug (1, 5) and (3,-1) into the distance formula to find the shortest distance.

$$\begin{aligned} d &= \sqrt{(3-1)^2+(-1-5)^2} \\ &= \sqrt{(2)^2+(-6)^2} \\ &= \sqrt{2+36} \\ &= \sqrt{38} \approx 6.16 \; units \end{aligned}$$

Shortest Distance between Two Parallel Lines

The shortest distance between two parallel lines is the length of the perpendicular segment between them. It doesn't matter which perpendicular line you choose, as long as the two points are on the lines. Recall that there are infinitely many perpendicular lines between two parallel lines.



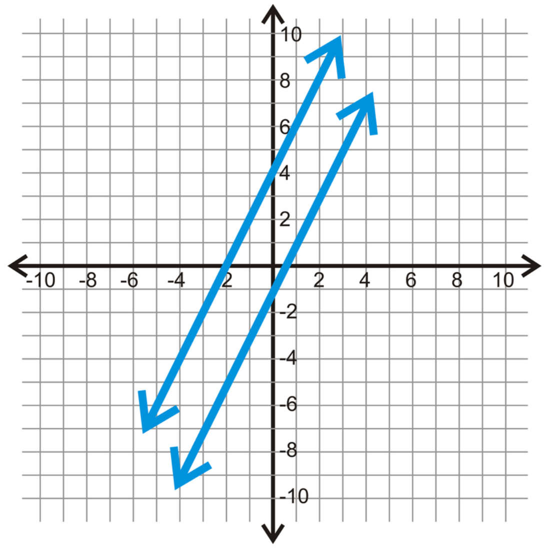
Notice that all of the pink segments are the same length. So, when picking a perpendicular segment, be sure to pick one with endpoints that are integers.

Example 3: Find the distance between $x = 3$ and $x = -5$.

Solution: Any line with $x = a$ number is a vertical line. In this case, we can just count the squares between the two lines. The two lines are $3-(-5)$ units apart, or 8 units.

You can use this same method with horizontal lines as well. For example, $y = -1$ and $y = 3$ are $3-(-1)$ units, or 4 units apart.

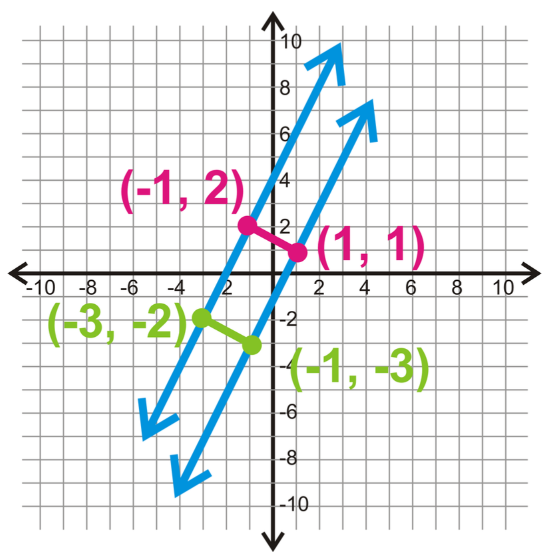
Example 4: What is the shortest distance between $y = 2x+4$ and $y = 2x-1$?



Solution: Graph the two lines and determine the perpendicular slope, which is $-\frac{1}{2}$. Find a point on $y = 2x+4$, let's say (-1, 2). From here, use the slope of the perpendicular line to find the corresponding point on $y = 2x-1$. If you move down 1 from 2 and over to the right 2 from -1, you will hit $y = 2x-1$ at (1, 1). Use these two points to determine the distance between the two lines.

$$\begin{aligned} d &= \sqrt{(1+1)^2+(1-2)^2} \\ &= \sqrt{2^2+(-1)^2} \\ &= \sqrt{4+1} \\ &= \sqrt{5} \approx 2.24 \ units \end{aligned}$$

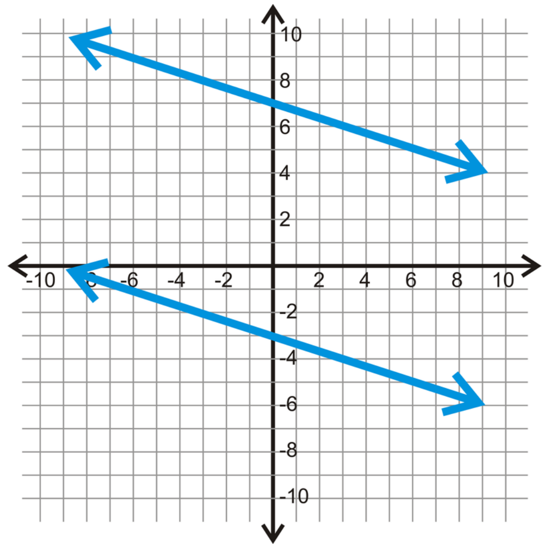
The lines are about 2.24 units apart.



Notice that you could have used any two points, as long as they are on the same perpendicular line. For example, you could have also used (-3, -2) and (-1, -3) and you still would have gotten the same answer.

$$\begin{aligned} d &= \sqrt{(-1+3)^2+(-3+2)^2} \\ &= \sqrt{2^2+(-1)^2} \\ &= \sqrt{4+1} \\ &= \sqrt{5} \approx 2.24 \ units \end{aligned}$$

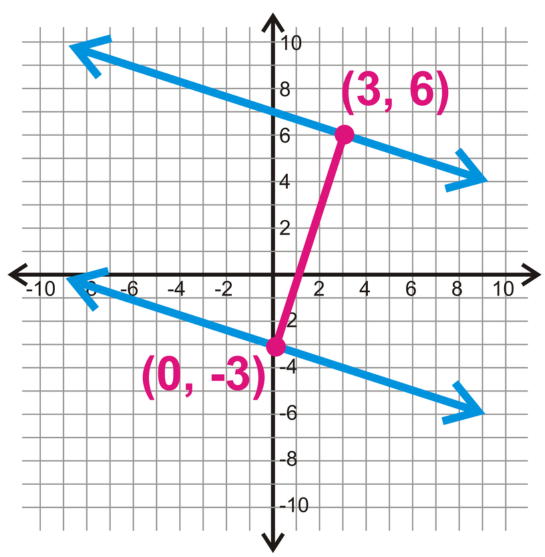
Example 5: Find the distance between the two parallel lines below.



Solution: First you need to find the slope of the two lines. Because they are parallel, they are the same slope, so if you find the slope of one, you have the slope of both.

Start at the $y-$intercept of the top line, 7. From there, you would go down 1 and over 3 to reach the line again. Therefore the slope is $-\frac{1}{3}$ and the perpendicular slope would be 3.

Next, find two points on the lines. Let's use the $y-$intercept of the bottom line, (0, -3). Then, rise 3 and go over 1 until your reach the second line. Doing this three times, you would hit the top line at (3, 6). Use these two points in the distance formula to find how far apart the lines are.

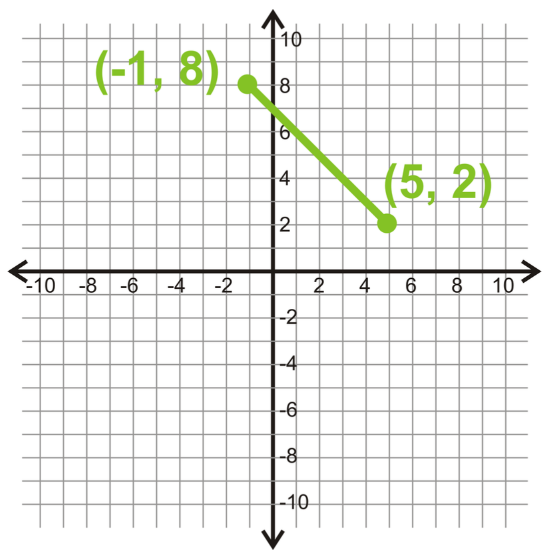


$$
\begin{aligned}
d &= \sqrt{(0-3)^2+(-3-6)^2} \\
&= \sqrt{(-3)^2+(-9)^2} \\
&= \sqrt{9+81} \\
&= \sqrt{90} \approx 9.49 \; units
\end{aligned}
$$

Perpendicular Bisectors in the Coordinate Plane

Recall that the definition of a perpendicular bisector is a perpendicular line that goes through the midpoint of a line segment. Using what we have learned in this chapter and the formula for a midpoint, we can find the equation of a perpendicular bisector.

Example 6: Find the equation of the perpendicular bisector of the line segment between (-1, 8) and (5, 2).



Solution: First, find the midpoint of the line segment.

$$\left(\frac{-1+5}{2}, \frac{8+2}{2}\right) = \left(\frac{4}{2}, \frac{10}{2}\right) = (2,\ 5)$$

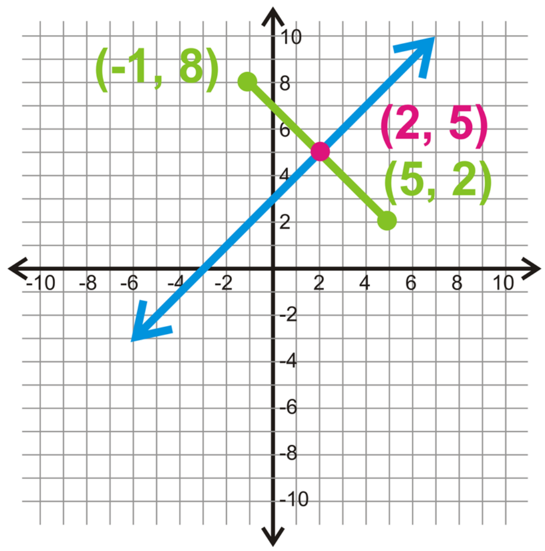
Second, find the slope between the two endpoints. This will help us figure out the perpendicular slope for the perpendicular bisector.

$$m = \frac{2-8}{5+1} = \frac{-6}{6} = -1$$

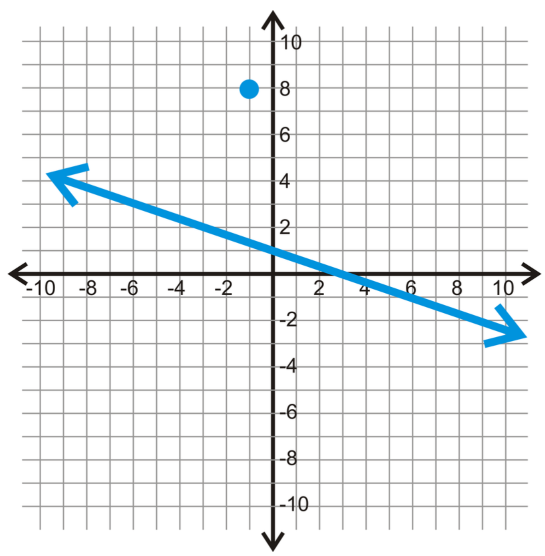
If the slope of the segment is -1, then the slope of the perpendicular bisector will be 1. The last thing to do is to find the $y-$intercept of the perpendicular bisector. We know it goes through the midpoint, (2, 5), of the segment, so substitute that in for x and y in the slope-intercept equation.

$$\begin{aligned}y &= mx+b\\5 &= 1(2)+b\\5 &= 2+b\\3 &= b\end{aligned}$$

The equation of the perpendicular bisector is $y = x+3$.



Example 7: The perpendicular bisector of $\overline{AB}$ has the equation $y = -\frac{1}{3}x+1$. If A is (-1, 8) what are the coordinates of B?



Solution: The easiest way to approach this problem is to graph it. Graph the perpendicular line and plot the point. See the graph to the left.

Second, determine the slope of $\overline{AB}$. If the slope of the perpendicular bisector is $-\frac{1}{3}$, then the slope of $\overline{AB}$ is 3.

Using the slope, count *down* 3 and over to the *right* 1 until you hit the perpendicular bisector. Counting down 6 and over 2, you land on the line at (-3, 2). This is the *midpoint* of $\overline{AB}$. If you count down another 6 and over to the right 2 more, you will find the coordinates of B, which are (-5, -4).

Know What? Revisited Draw two intersecting lines. Make sure they are not perpendicular. Label the 26.3 miles along hwy 5. The longest distance is found by adding the distances along the 110 and 405, or 41.8 miles.

Review Questions

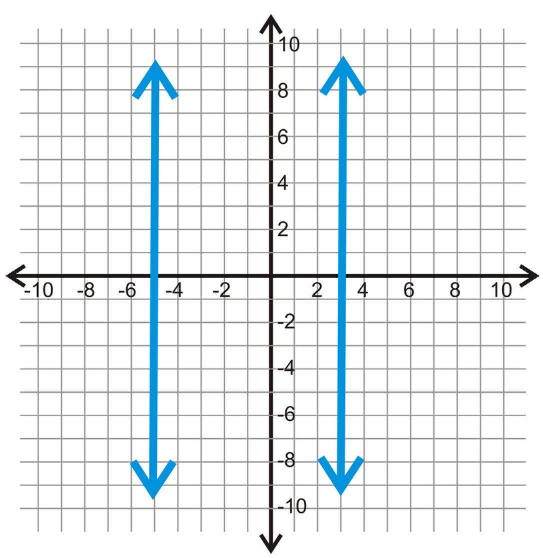
Find the distance between each pair of points. Round your answer to the nearest hundredth.

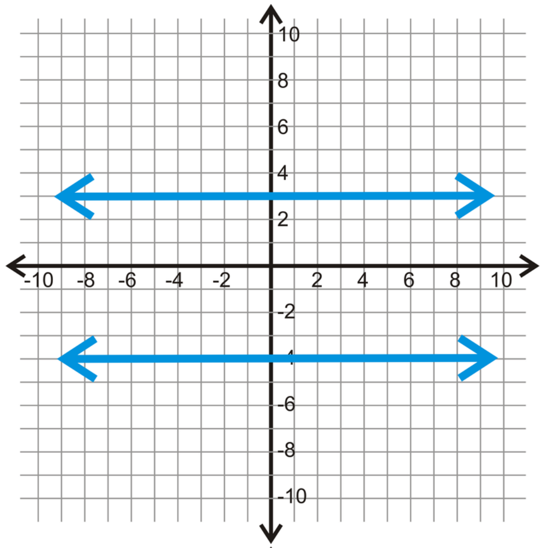
1. (4, 15) and (-2, -1)
2. (-6, 1) and (9, -11)
3. (0, 12) and (-3, 8)
4. (-8, 19) and (3, 5)
5. (3, -25) and (-10, -7)
6. (-1, 2) and (8, -9)
7. (5, -2) and (1, 3)
8. (-30, 6) and (-23, 0)

Determine the shortest distance between the given line and point. Round your answers to the nearest hundredth.

9. $y = \frac{1}{3}x + 4;\ (5,\ -1)$
10. $y = 2x - 4;\ (-7,\ -3)$
11. $y = -4x + 1;\ (4,\ 2)$
12. $y = -\frac{2}{3}x - 8;\ (7,\ 9)$

Use each graph below to determine how far apart each theparallel lines are. Round your answers to the nearest hundredth.

13. 



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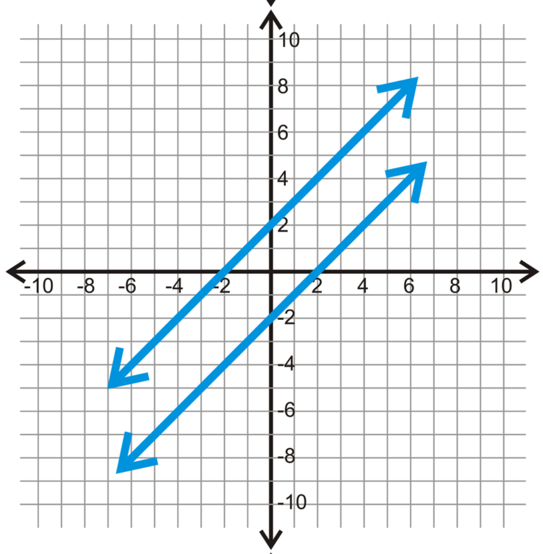
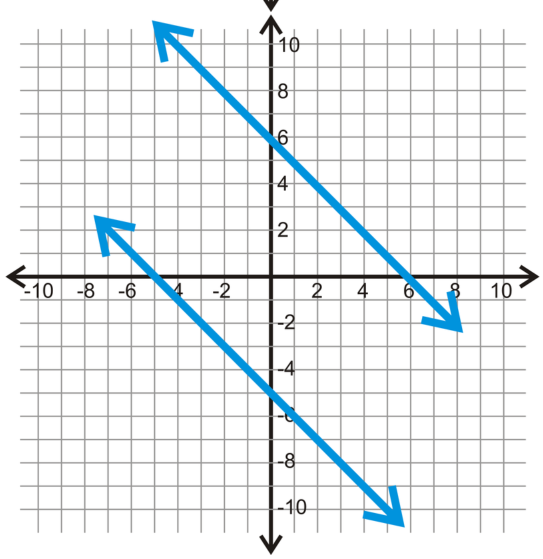


15.



16.

Determine the shortest distance between the each pair of parallel lines. Round your answer to the nearest hundredth.

17. $x = 5, \ x = 1$
18. $y = -6, \ y = 4$
19. $y = x + 5, \ y = x - 3$

20. $y = -\frac{1}{3}x + 2,\ y = -\frac{1}{3}x - 8$
21. $y = 4x + 9,\ y = 4x - 8$
22. $y = \frac{1}{2}x,\ y = \frac{1}{2}x - 5$

Find the equation of the perpendicular bisector for pair of points.

23. (1, 5) and (7, -7)
24. (1, -8) and (7, -6)
25. (9, 2) and (-9, -10)
26. (-7, 11) and (-3, 1)
27. The perpendicular bisector of $\overline{CD}$ has the equation $y = 3x - 11$. If D is (-3, 0) what are the coordinates of C?
28. The perpendicular bisector of $\overline{LM}$ has the equation $y = -x + 5$. If L is (6, -3) what are the coordinates of M?
29. ***Construction*** Plot the points (5, -3) and (-5, -9). Draw the line segment between the points. Construct the perpendicular bisector for these two points. (Construction was in Chapter 1). Determine the equation of the perpendicular bisector and the midpoint.
30. ***Construction*** Graph the line $y = -\frac{1}{2}x - 5$ and the point (2, 5). Construct the perpendicular line, through (2, 5) and determine the equation of this line.
31. ***Challenge*** The distance between two points is 25 units. One point is (-2, 9). What is the second point? You may assume that the second point is made up of integers.
32. ***Writing*** List the steps you would take to find the distance between two parallel lines, like the two in #24.

Review Queue Answers

1. $y = -4x - 1$
2. $y = \frac{1}{2}x - 3$
3. $y = \frac{2}{3}x + 6$

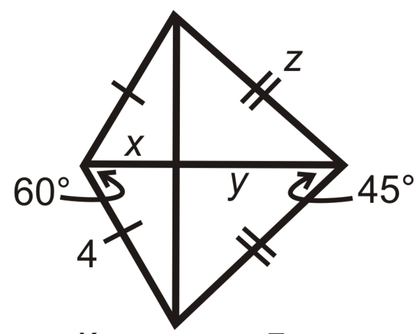
1.7 Tangent, Sine and Cosine

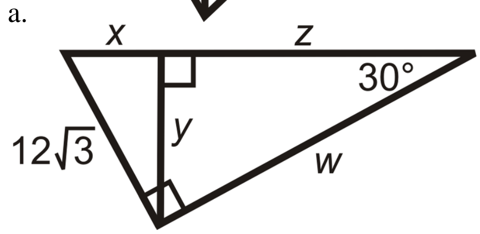
Learning Objectives

- Use the tangent, sine and cosine ratios in a right triangle.
- Understand these trigonometric ratios in special right triangles.
- Use a scientific calculator to find sine, cosine and tangent.
- Use trigonometric ratios in real-life situations.

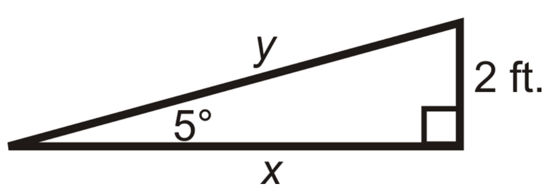
Review Queue

1. The legs of an isosceles right triangle have length 14. What is the hypotenuse?
2. Do the lengths 8, 16, 20 make a right triangle? If not, is the triangle obtuse or acute?
3. In a 30-60-90 triangle, what do the 30, 60, and 90 refer to?
4. Find the measure of the missing lengths.

a. 

b. 

Know What? A restaurant needs to build a wheelchair ramp for its customers. The angle of elevation for a ramp is recommended to be 5°. If the vertical distance from the sidewalk to the front door is two feet, what is the horizontal distance that the ramp will take up (x)? How long will the ramp be (y)? Round your answers to the nearest hundredth.



What is Trigonometry?

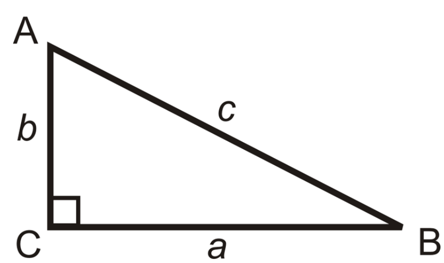
The word trigonometry comes from two words meaning *triangle* and *measure*. In this lesson we will define three trigonometric (or trig) functions. Once we have defined these functions, we will be able to solve problems like the **Know What?** above.

Trigonometry: The study of the relationships between the sides and angles of right triangles.

In trigonometry, sides are named in reference to a particular angle. The hypotenuse of a triangle is always the same, but the terms **adjacent** and **opposite** depend on which angle you are referencing. A side adjacent to an angle is the leg of the triangle that helps form the angle. A side opposite to an angle is the leg of the triangle that does not help form the angle. We never reference the right angle when referring to trig ratios.

a is $adjacent$ to $\angle B$. a is $opposite$ $\angle A$.
b is $adjacent$ to $\angle A$. b is $opposite$ $\angle B$.

c is the $hypotenuse$.



Sine, Cosine, and Tangent Ratios

The three basic trig ratios are called, sine, cosine and tangent. At this point, we will only take the sine, cosine and tangent of acute angles. However, you will learn that you can use these ratios with obtuse angles as well.

Sine Ratio: For an acute angle x in a right triangle, the $\sin x$ is equal to the ratio of the side opposite the angle over the hypotenuse of the triangle.

Using the triangle above, $\sin A = \frac{a}{c}$ and $\sin B = \frac{b}{c}$.

Cosine Ratio: For an acute angle x in a right triangle, the $\cos x$ is equal to the ratio of the side adjacent to the angle over the hypotenuse of the triangle.

Using the triangle above, $\cos A = \frac{b}{c}$ and $\cos B = \frac{a}{c}$.

Tangent Ratio: For an acute angle x, in a right triangle, the $\tan x$ is equal to the ratio of the side opposite to the angle over the side adjacent to x.

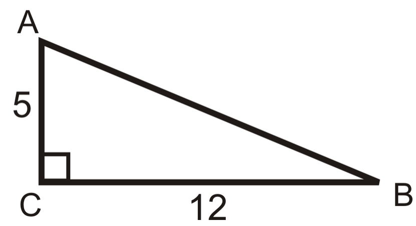
Using the triangle above, $\tan A = \frac{a}{b}$ and $\tan B = \frac{b}{a}$.

There are a few important things to note about the way we write these ratios. First, keep in mind that the abbreviations $\sin x, \cos x$, and $\tan x$ are all functions. Each ratio can be considered a function of the angle (see Chapter 10). Second, be careful when using the abbreviations that you still pronounce the full name of each function. When we write $\sin x$ it is still pronounced *sine,* with a long "*i*". When we write $\cos x$, we still say co-sine. And when we write $\tan x$, we still say tangent.

An easy way to remember ratios is to use the pneumonic SOH-CAH-TOA.

$$\text{Sine} = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \text{Cosine} = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \text{Tangent} = \frac{\text{Opposite}}{\text{Adjacent}}$$

Example 1: Find the sine, cosine and tangent ratios of $\angle A$.



Solution: First, we need to use the Pythagorean Theorem to find the length of the hypotenuse.

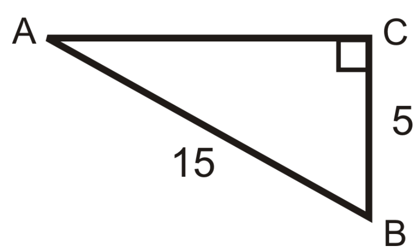
$$5^2 + 12^2 = h^2$$
$$13 = h$$

So, $\sin A = \frac{12}{13}, \cos A =$, and $\tan A = \frac{12}{5}$.

A few important points:

- Always reduce ratios when you can.
- Use the Pythagorean Theorem to find the missing side (if there is one).
- The tangent ratio can be bigger than 1 (the other two cannot).
- If two right triangles are similar, then their sine, cosine, and tangent ratios will be the same (because they will reduce to the same ratio).
- If there is a radical in the denominator, rationalize the denominator.

Example 2: Find the sine, cosine, and tangent of $\angle B$.

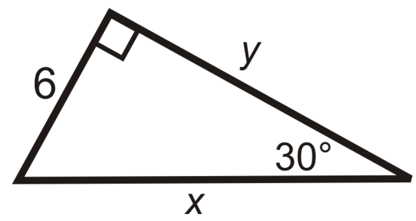


Solution: Find the length of the missing side.

$$AC^2 + 5^2 = 15^2$$
$$AC^2 = 200$$
$$AC = 10\sqrt{2}$$

Therefore, $\sin B = \frac{10\sqrt{2}}{15} = \frac{2\sqrt{2}}{3}, \cos B = \frac{5}{15} = \frac{1}{3}$, and $\tan B = \frac{10\sqrt{2}}{5} = 2\sqrt{2}$.

Example 3: Find the sine, cosine and tangent of 30°.



Solution: This is a special right triangle, a 30-60-90 triangle. So, if the short leg is 6, then the long leg is $6\sqrt{3}$ and the hypotenuse is 12.

$\sin 30^\circ = \frac{6}{12} = \frac{1}{2}, \cos 30^\circ = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2}$, and $\tan 30^\circ = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

In Example 3, we knew the angle measure of the angle we were taking the sine, cosine and tangent of. This means that ***the sine, cosine and tangent for an angle are fixed***.

Sine, Cosine, and Tangent with a Calculator

We now know that the trigonometric ratios are not dependent on the sides, but the ratios. Therefore, there is one fixed value for every angle, from 0° to 90°. Your scientific (or graphing) calculator knows the values of the sine, cosine and tangent of all of these angles. Depending on your calculator, you should have [SIN], [COS], and [TAN] buttons. Use these to find the sine, cosine, and tangent of any acute angle.

Example 4: Find the indicated trigonometric value, using your calculator.

a) $\sin 78^\circ$

b) $\cos 60^\circ$

c) $\tan 15^\circ$

Solution: Depending on your calculator, you enter the degree first, and then press the correct trig button or the other way around. For TI-83s and TI-84s you press the trig button first, followed by the angle. Also, make sure the mode of your calculator is in DEGREES.

a) $\sin 78^\circ = 0.9781$

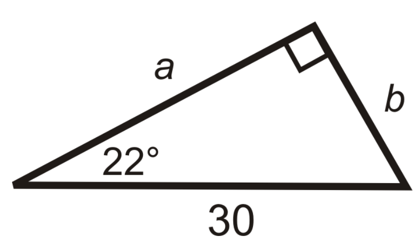
b) $\cos 60^\circ = 0.5$

c) $\tan 15^\circ = 0.2679$

Finding the Sides of a Triangle using Trig Ratios

One application of the trigonometric ratios is to use them to find the missing sides of a right triangle. All you need is one angle, other than the right angle, and one side. Let's go through a couple of examples.

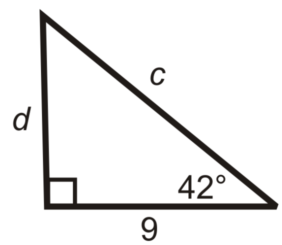
Example 5: Find the value of each variable. Round your answer to the nearest hundredth.



Solution: We are given the hypotenuse, so we would need to use the sine to find b, because it is opposite 22° and cosine to find a, because it is adjacent to 22°.

$$\sin 22^\circ = \frac{b}{30} \qquad \cos 22^\circ = \frac{a}{30}$$
$$30 \cdot \sin 22^\circ = b \qquad 30 \cdot \cos 22^\circ = a$$
$$b \approx 11.24 \qquad a \approx 27.82$$

Example 6: Find the value of each variable. Round your answer to the nearest hundredth.



Solution: Here, we are given the adjacent leg to 42°. To find c, we need to use cosine and to find d we will use tangent.

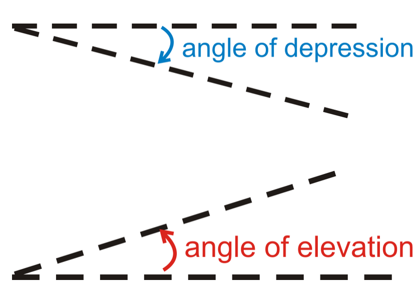
$$\cos 42^\circ = \frac{9}{c} \qquad \tan 42^\circ = \frac{d}{9}$$
$$c \cdot \cos 42^\circ = 9 \qquad 9 \cdot \tan 42^\circ = d$$
$$c = \frac{9}{\cos 42^\circ} \approx 12.11 \qquad d \approx 8.10$$

Notice in both of these examples, you should only use the information that you are given. For example, you should not use the found value of b to find a (in Example 5) because b is an ***approximation***. Use exact values to give the most accurate answers. However, in both examples you could have also used the complementary angle to the one given.

Angles of Depression and Elevation

Another practical application of the trigonometric functions is to find the measure of lengths that you cannot measure. Very frequently, angles of depression and elevation are used in these types of problems.

Angle of Depression: The angle measured from the horizon or horizontal line, down.



Angle of Elevation: The angle measure from the horizon or horizontal line, up.

Example 7: An inquisitive math student is standing 25 feet from the base of the Washington Monument. The angle of elevation from her horizontal line of sight is 87.4°. If her "eye height" is 5ft, how tall is the monument?

Solution: We can find the height of the monument by using the tangent ratio and then adding the eye height of the student.

$$\tan 87.4^\circ = \frac{h}{25}$$
$$h = 25 \cdot \tan 87.4^\circ = 550.54$$

Adding 5 ft, the total height of the Washington Monument is 555.54 ft.

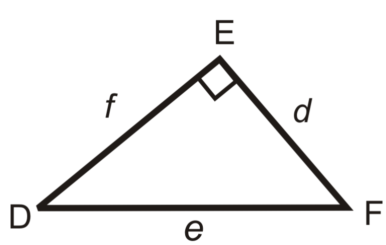
According to Wikipedia, the actual height of the monument is 555.427 ft.

Know What? Revisited To find the horizontal length and the actual length of the ramp, we need to use the tangent and sine.

$$\tan 5^\circ = \frac{2}{x} \qquad \sin 5^\circ = \frac{2}{y}$$
$$x = \frac{2}{\tan 5^\circ} = 22.86 \qquad y = \frac{2}{\sin 5^\circ} = 22.95$$

Review Questions

Use the diagram to fill in the blanks below.



1. $\tan D = \frac{?}{?}$
2. $\sin F = \frac{?}{?}$
3. $\tan F = \frac{?}{?}$
4. $\cos F = \frac{?}{?}$
5. $\sin D = \frac{?}{?}$
6. $\cos D = \frac{?}{?}$

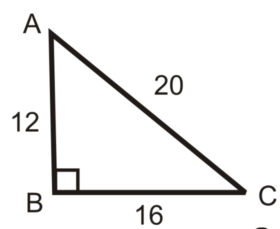
From questions 1-6, we can conclude the following. Fill in the blanks.

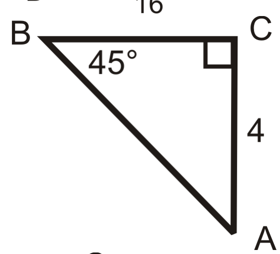
7. $\cos \_\_\_ = \sin F$ and $\sin \_\_\_ = \cos F$
8. The sine of an angle is ____________ to the cosine of its ______________.
9. $\tan D$ and $\tan F$ are ___________ of each other.

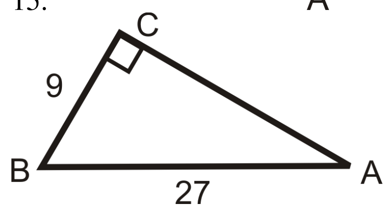
Use your calculator to find the value of each trig function below. Round to four decimal places.

10. $\sin 24^\circ$
11. $\cos 45^\circ$
12. $\tan 88^\circ$
13. $\sin 43^\circ$

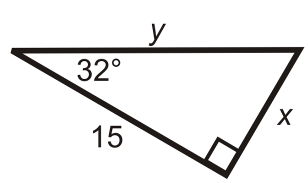
Find the sine, cosine and tangent of $\angle A$. Reduce all fractions and radicals.

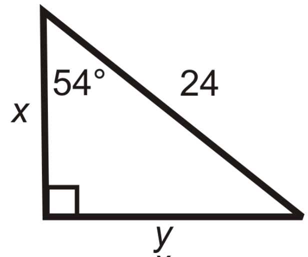
14. 

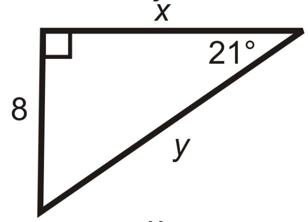
15. 

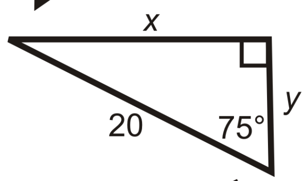
16. 

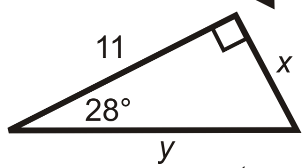
Find the length of the missing sides. Round your answers to the nearest hundredth.

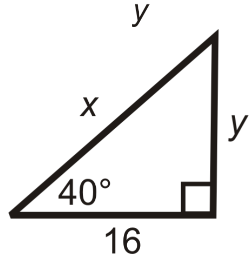
17. 

18. 

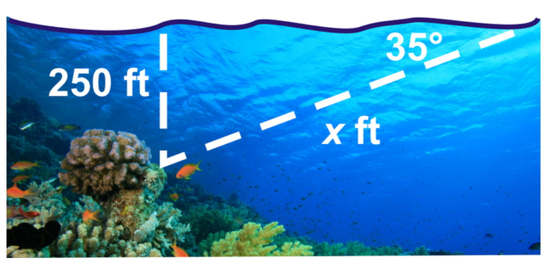
19. 

20. 

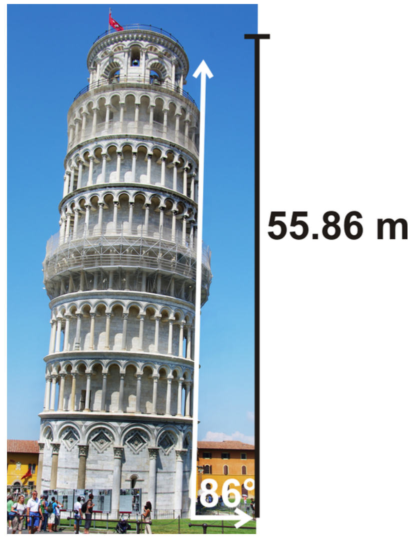
21. 

22. 

23. Kristin is swimming in the ocean and notices a coral reef below her. The angle of depression is 35° and the depth of the ocean, at that point, is 250 feet. How far away is she from the reef?



24. The Leaning Tower of Piza currently "leans" at a 4° angle and has a vertical height of 55.86 meters. How tall was the tower when it was originally built?



25. The angle of depression from the top of an apartment building to the base of a fountain in a nearby park is 72°. If the building is 78 ft tall, how far away is the fountain?
26. William spots a tree directly across the river from where he is standing. He then walks 20 ft upstream and determines that the angle between his previous position and the tree on the other side of the river is 65°. How wide is the river?
27. Diego is flying his kite one afternoon and notices that he has let out the entire 120 ft of string. The angle his string makes with the ground is 52°. How high is his kite at this time?
28. A tree struck by lightning in a storm breaks and falls over to form a triangle with the ground. The tip of the tree makes a 36° angle with the ground 25 ft from the base of the tree. What was the height of the tree to the nearest foot?
29. Upon descent an airplane is 20,000 ft above the ground. The air traffic control tower is 200 ft tall. It is determined that the angle of elevation from the top of the tower to the plane is 15°. To the nearest mile, find the ground distance from the airplane to the tower.
30. ***Critical Thinking*** Why are the sine and cosine ratios always be less than 1?

Review Queue Answers

1. The hypotenuse is $14\sqrt{2}$.
2. No, $8^2 + 16^2 < 20^2$, the triangle is obtuse.
3. $30^\circ, 60^\circ$, and 90° refer to the angle measures in the special right triangle.
4. $x = 2, y = 2\sqrt{3}$
5. $x = 6\sqrt{3}, y = 18, z = 18\sqrt{3}, w = 36$

1.8 Right Triangle Trigonometry

Objective

To develop an understanding of trigonometric ratios and to use the trigonometric ratios sine, cosine and tangent along with their inverses to solve right triangles.

Review Queue

1. Given that $P(A) = 0.8, P(B) = 0.5$ and $P(A \cup B) = 0.9$, determine whether events A and B are independent.

2. Events A and B are independent and $P(A) = 0.6$ and $P(B) = 0.5$, find $P(A \cup B)'$.

3. Reduce the radical expressions:

a. $\sqrt{240}$

b. $3\sqrt{48} + 5\sqrt{75}$

c. $4\sqrt{15} \cdot \sqrt{30}$

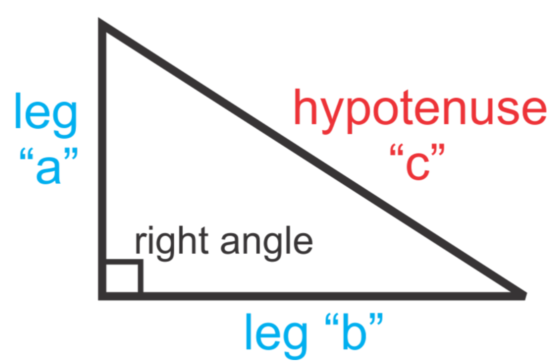
Pythagorean Theorem and its Converse

Objective

Discover, prove and apply the Pythagorean Theorem to solve for unknown sides in right triangles and prove triangles are right triangles.

Guidance

The Pythagorean Theorem refers to the relationship between the lengths of the three sides in a right triangle. It states that if a and b are the legs of the right triangle and c is the hypotenuse, then $a^2 + b^2 = c^2$. For example, the lengths 3, 4, and 5 are the sides of a right triangle because $3^2 + 4^2 = 5^2 (9 + 16 = 25)$. Keep in mind that c is always the longest side.

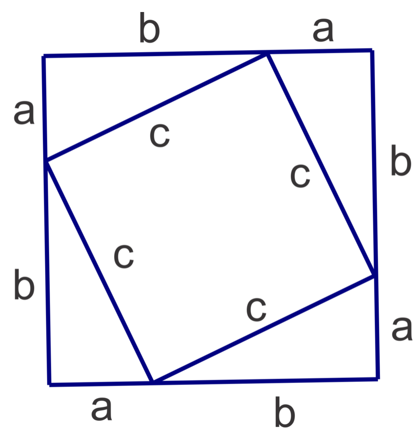


The converse of this statement is also true. If, in a triangle, c is the length of the longest side and the shorter sides have lengths a and b, and $a^2 + b^2 = c^2$, then the triangle is a right triangle.

Proof of Pythagorean Theorem

There are many proofs of the Pythagorean Theorem and here is one of them. We will be using the concept that the area of a figure is equal to the sum of the areas of the smaller figures contained within it and algebra to derive the Pythagorean Theorem.

Using the figure below (a square with a smaller square inside), first write two equations for its area, one using the lengths of the sides of the outer square and one using the sum of the areas of the smaller square and the four triangles.



Area 1: $(a+b)^2 = a^2 + 2ab + b^2$

Area 2: $c^2 + 4\left(\frac{1}{2}ab\right) = c^2 + 2ab$

Now, equate the two areas and simplify:

$$a^2 + 2ab + b^2 = c^2 + 2ab$$
$$a^2 + b^2 = c^2$$

Example A

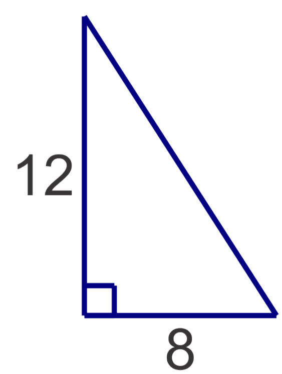
In a right triangle $a = 7$ and $c = 25$, find the length of the third side.

Solution: We can start by substituting what we know into the Pythagorean Theorem and then solve for the unknown side, b:

$$\begin{aligned} 7^2 + b^2 &= 25^2 \\ 49 + b^2 &= 625 \\ b^2 &= 576 \\ b &= 24 \end{aligned}$$

Example B

Find the length of the third side of the triangle below. Leave your answer in reduced radical form.



Solution: Since we are given the lengths of the two legs, we can plug them into the Pythagorean Theorem and find the length of the hypotenuse.

$$\begin{aligned}
8^2 + 12^2 &= c^2\\
64 + 144 &= c^2\\
c^2 &= 208\\
c &= \sqrt{208} = \sqrt{16 \cdot 13} = 4\sqrt{13}
\end{aligned}$$

Example C

Determine whether a triangle with lengths 21, 28, 35 is a right triangle.

Solution: We need to see if these values will satisfy $a^2 + b^2 = c^2$. If they do, then a right triangle is formed. So,

$$\begin{aligned}
21^2 + 28^2 &= 441 + 784 = 1225\\
35^2 &= 1225
\end{aligned}$$

Yes, the Pythagorean Theorem is satisfied by these lengths and a right triangle is formed by the lengths 21, 28 and 35.

Guided Practice

For the given two sides, determine the length of the third side if the triangle is a right triangle.

1. $a = 10$ and $b = 5$

2. $a = 5$ and $c = 13$

Use the Pythagorean Theorem to determine if a right triangle is formed by the given lengths.

3. 16, 30, 34

4. 9, 40, 42

5. 2, 2, 4

Answers

1. $\sqrt{10^2 + 5^2} = \sqrt{100 + 25} = \sqrt{125} = 5\sqrt{5}$

2. $\sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12$

3.

$$\begin{aligned}
16^2 + 30^2 &= 256 + 900 = 1156\\
34^2 &= 1156
\end{aligned}$$

Yes, this is a right triangle.

4.

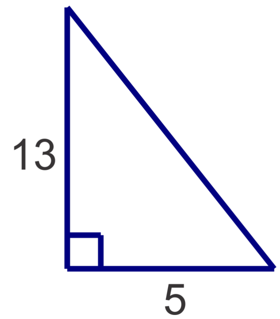
$$\begin{aligned}
9^2 + 40^2 &= 81 + 1600 = 1681\\
42^2 &= 1764
\end{aligned}$$

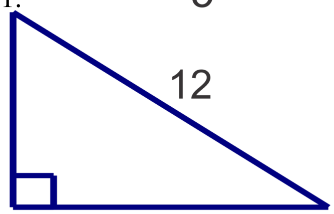
No, this is not a right triangle.

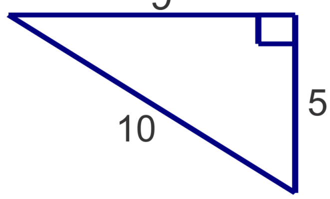
5. This one is tricky, in a triangle the lengths of any two sides must have a sum *greater* than the length of the third side. These lengths do not meet that requirement so not only do they not form a *right* triangle, they do not make a triangle at all.

Problem Set

Find the unknown side length for each right triangle below.

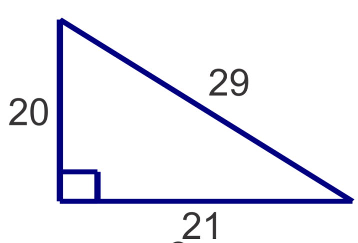
1. 

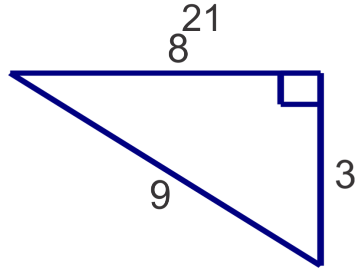
2. 

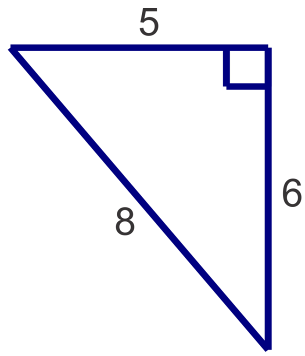
3. 

4. $a = 6, b = 8$
5. $b = 6, c = 14$
6. $a = 12, c = 18$

Determine whether the following triangles are right triangles.

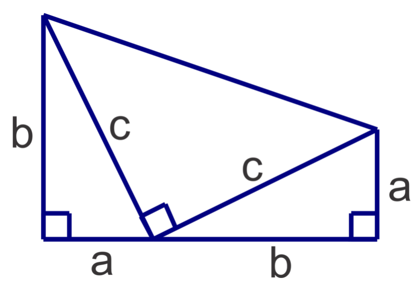
7. 

8. 

9. 

Do the lengths below form a right triangle? Remember to make sure that they form a triangle.

10. 3, 4, 5
11. 6, 6, 11
12. 11, 13, 17
13. Major General James A. Garfield (and former President of the U.S.) is credited with deriving this proof of the Pythagorean Theorem using a trapezoid. Follow the steps to recreate his proof.



(a) Find the area of the trapezoid using the trapezoid area formula: $A = \frac{1}{2}(b_1 + b_2)h$ (b) Find the sum of the areas of the three right triangles in the diagram. (c) The areas found in the previous two problems should be the same value. Set the expressions equal to each other and simplify to get the Pythagorean Theorem.

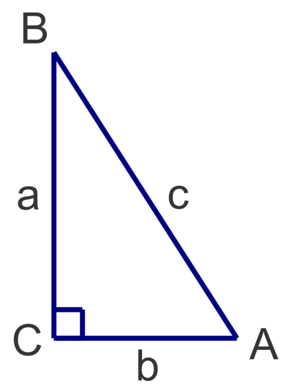
Sine, Cosine, and Tangent

Objective

Define and apply the trigonometric ratios sine, cosine and tangent to solve for the lengths of unknown sides in right triangles.

Guidance

The trigonometric ratios sine, cosine and tangent refer to the known ratios between particular sides in a right triangle based on an acute angle measure.



In this right triangle, side c is the hypotenuse.

If we consider the angle B, then we can describe each of the legs by its position relative to angle B: side a is adjacent to B; side b is opposite B

If we consider the angle A, then we can describe each of the legs by its position relative to angle A: side b is adjacent to A; side a is opposite A

Now we can define the trigonometry ratios as follows:

$$\text{Sine is } \frac{opposite}{hypotenuse} \quad \text{Cosine is } \frac{adjacent}{hypotenuse} \quad \text{Tangent is } \frac{opposite}{adjacent}$$

A shorthand way to remember these ratios is to take the letters in red above and write the phrase:

SOH CAH TOA

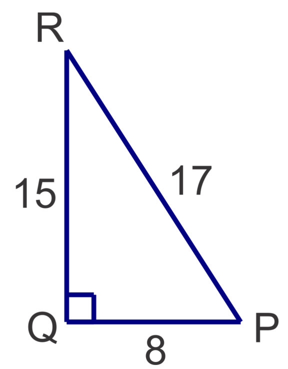
Now we can find the trigonometric ratios for each of the acute angles in the triangle above.

$$\begin{aligned} \sin A &= \frac{a}{c} & \sin B &= \frac{b}{c} \\ \cos A &= \frac{b}{c} & \cos B &= \frac{a}{c} \\ \tan A &= \frac{a}{b} & \tan B &= \frac{b}{a} \end{aligned}$$

It is important to understand that given a particular (acute) angle measure in a right triangle, these ratios are constant no matter how big or small the triangle. For example; if the measure of the angle is 25°, then $\sin 25^\circ \approx 0.4226$ and ratio of the opposite side to the hypotenuse is always 0.4226 no matter how big or small the triangle.

Example A

Find the trig ratios for the acute angles R and P in ΔPQR.



Solution: From angle R, $O = 8$; $A = 15$; and $H = 17$. Now the trig ratios are:

$$\sin R = \frac{8}{17}; \ \cos R = \frac{15}{17}; \ \tan R = \frac{8}{15}$$

From angle P, $O = 15$; $A = 8$; and $H = 17$. Now the trig ratios are:

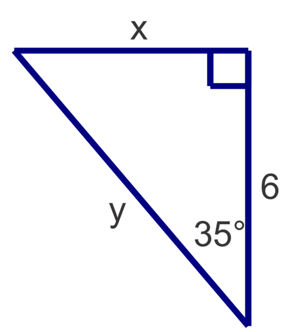
$$\sin P = \frac{15}{17}; \ \cos P = \frac{8}{17}; \ \tan P = \frac{15}{8}$$

Do you notice any patterns or similarities between the trigonometric ratios? The opposite and adjacent sides are switched and the hypotenuse is the same. Notice how this switch affects the ratios:

$$\sin R = \cos P \quad \cos R = \sin P \quad \tan R = \frac{1}{\tan P}$$

Example B

Use trigonometric ratios to find the x and y.



Solution: First identify or label the sides with respect to the given acute angle. So, x is opposite, y is hypotenuse (note that it is the hypotenuse because it is the side opposite the right angle, it may be adjacent to the given angle but the hypotenuse cannot be the adjacent side) and 6 is the adjacent side.

To find x, we must use the given length of 6 in our ratio too. So we are using opposite and adjacent. Since tangent is the ratio of opposite over adjacent we get:

$$\begin{aligned}
\tan 35^\circ &= \frac{x}{6} \\
x &= 6 \tan 35^\circ && \text{multiply both sides by 6} \\
x &\approx 4.20 && \text{Use the calculator to evaluate-type in 6TAN(35) ENTER}
\end{aligned}$$

NOTE: make sure that your calculator is in DEGREE mode. To check, press the MODE button and verify that DEGREE is highlighted (as opposed to RADIAN). If it is not, use the arrow buttons to go to DEGREE and press ENTER. The default mode is radian, so if your calculator is reset or the memory is cleared it will go back to radian mode until you change it.

To find y using trig ratios and the given length of 6, we have adjacent and hypotenuse so we'll use cosine:

$$\begin{aligned}
\cos 35^\circ &= \frac{6}{y} \\
\frac{\cos 35^\circ}{1} &= \frac{6}{y} && \text{set up a proportion to solve for } y \\
6 &= y\cos 35^\circ && \text{cross multiply} \\
y &= \frac{6}{\cos 35^\circ} && \text{divide by} \cos 35^\circ \\
y &= 7.32 && \text{Use the calculator to evaluate-type in 6/TAN(35) ENTER}
\end{aligned}$$

Alternatively, we could find y using the value we found for x and the Pythagorean theorem:

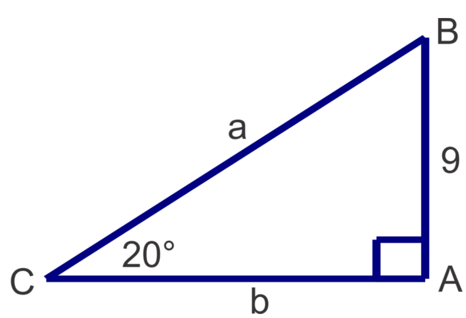
$$\begin{aligned}
4.20^2 + 6^2 &= y^2 \\
53.64 &= y^2 \\
y &\approx 7.32
\end{aligned}$$

The downside of this method is that if we miscalculated our x value, we will double down on our mistake and guarantee an incorrect y value. In general you will help avoid this kind of mistake if you use the given information whenever possible.

Example C

Given ΔABC, with $m\angle A = 90^\circ, m\angle C = 20^\circ$ and $c = 9$, find a and b.

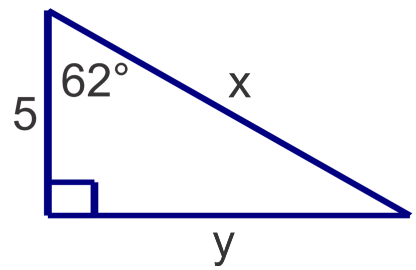
Solution: Visual learners may find it particularly useful to make a sketch of this triangle and label it with the given information:



To find a (the hypotenuse) we can use the opposite side and the sine ratio: $\sin 20^\circ = \frac{9}{a}$, solving as we did in Example B we get $a = \frac{9}{\sin 20^\circ} \approx 26.31$ To find b (the adjacent side) we can use the opposite side and the tangent ratio: $\tan 20^\circ = \frac{9}{b}$, solving for b we get $b = \frac{9}{\tan 20^\circ} \approx 24.73$.

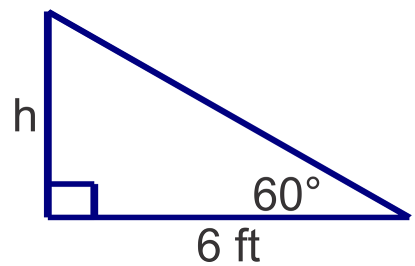
Guided Practice

1. Use trig ratios to find x and y:



2. Given ΔABC with $m\angle B = 90^\circ, m\angle A = 43^\circ$ and $a = 7$, find b and c.

3. The base of a playground slide is 6 ft from the base of the platform and the slide makes a 60° angle with the ground. To the nearest tenth of a foot, how high is the platform at the top of the slide?



Answers

1. For x:

$$\cos 62^\circ = \frac{5}{x}$$
$$x = \frac{5}{\cos 62^\circ} \approx 10.65$$

For y:

$$\tan 62^\circ = \frac{y}{5}$$
$$y = 5\tan 62^\circ \approx 9.40$$

2. For b:

$$\sin 43^\circ = \frac{7}{b}$$
$$b = \frac{7}{\sin 43^\circ} \approx 10.26$$

For c:

$$\tan 43^\circ = \frac{7}{c}$$
$$c = \frac{7}{\tan 43^\circ} \approx 7.51$$

3.

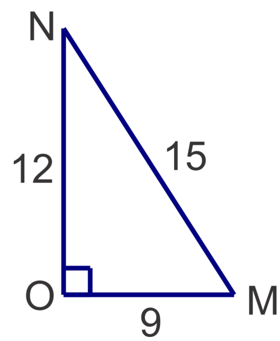
$$\tan 60^\circ = \frac{h}{6}$$
$$h = 6\tan 60^\circ \approx 10.39$$

, so the height of the platform is 10.4 ft

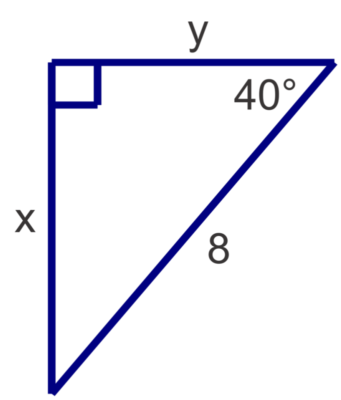
Problem Set

Use you calculator to find the following trigonometric ratios. Give answers to four decimal places.

1. $\sin 35^\circ$
2. $\tan 72^\circ$
3. $\cos 48^\circ$
4. Write the three trigonometric ratios of each of the acute angles in the triangle below.



Use trigonometric ratios to find the unknown side lengths in the triangles below. Round your answers to the nearest hundredth.



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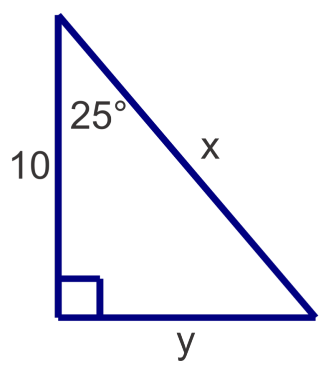

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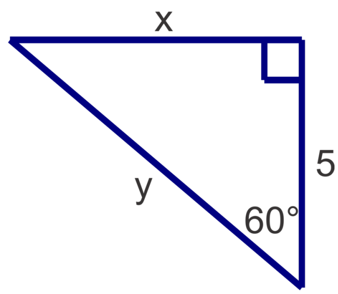
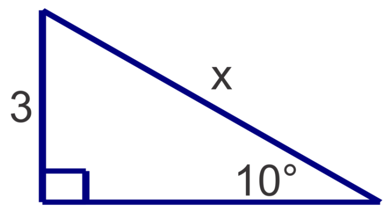


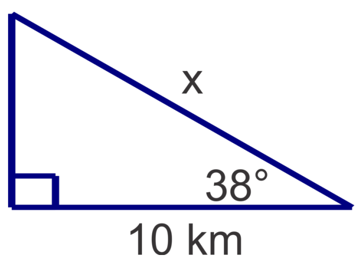
7.

For problems 8-10 use the given information about ΔABC with right angle B to find the unknown side lengths. Round your answer to the nearest hundredth.

8. $a = 12$ and $m\angle A = 43^\circ$
9. $m\angle C = 75^\circ$ and $b = 24$
10. $c = 7$ and $m\angle A = 65^\circ$
11. A ramp needs to have an angle of elevation no greater than 10 degrees. If the door is 3 ft above the sidewalk level, what is the minimum possible ramp length to the nearest tenth of a foot?



12. A ship, *Sea Dancer*, is 10 km due East of a lighthouse. A second ship, *Nelly*, is due north of the lighthouse. A spotter on the *Sea Dancer* measures the angle between the *Nelly* and the lighthouse to be 38°. How far apart are the two ships to the nearest tenth of a kilometer?



Inverse Trig Functions and Solving Right Triangles

Objective

Use the inverse trigonometric functions to find the measure of unknown acute angles in right triangles and solve right triangles.

Guidance

In the previous concept we used the trigonometric functions sine, cosine and tangent to find the ratio of particular sides in a right triangle given an angle. In this concept we will use the inverses of these functions, $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, to find the angle measure when the ratio of the side lengths is known. When we type $\sin 30^\circ$ into our calculator, the calculator goes to a table and finds the trig ratio associated with 30°, which is $\frac{1}{2}$. When we use an inverse function we tell the calculator to look up the ratio and give us the angle measure. For example: $\sin^{-1}\left(\frac{1}{2}\right) = 30^\circ$. On your calculator you would press 2^{ND}SIN to get $\text{SIN}^{-1}($ and then type in $\frac{1}{2}$, close the parenthesis and press ENTER. Your calculator screen should read $\text{SIN}^{-1}\left(\frac{1}{2}\right)$ when you press ENTER.

Example A

Find the measure of angle A associated with the following ratios. Round answers to the nearest degree.

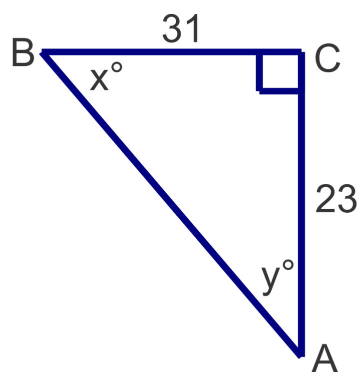
1. $\sin A = 0.8336$
2. $\tan A = 1.3527$
3. $\cos A = 0.2785$

Solution: Using the calculator we get the following:

1. $\sin^{-1}(0.8336) \approx 56^\circ$
2. $\tan^{-1}(1.3527) \approx 54^\circ$
3. $\cos^{-1}(0.2785) \approx 74^\circ$

Example B

Find the measures of the unknown angles in the triangle shown. Round answers to the nearest degree.



Solution: We can solve for either x or y first. If we choose to solve for x first, the 23 is opposite and 31 is adjacent so we will use the tangent ratio.

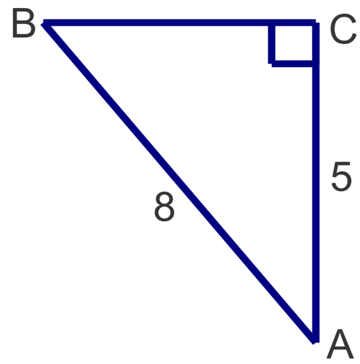
$$x = \tan^{-1}\left(\frac{23}{31}\right) \approx 37^\circ.$$

Recall that in a right triangle, the acute angles are always complementary, so $90^\circ - 37^\circ = 53^\circ$, so $y = 53^\circ$. We can also use the side lengths an a trig ratio to solve for y:

$$y = \tan^{-1}\left(\frac{31}{23}\right) \approx 53^\circ.$$

Example C

Solve the right triangle shown below. Round all answers to the nearest tenth.



Solution: We can solve for either angle A or angle B first. If we choose to solve for angle B first, then 8 is the hypotenuse and 5 is the opposite side length so we will use the sine ratio.

$$\sin B = \frac{5}{8}$$
$$m\angle B = \sin^{-1}\left(\frac{5}{8}\right) \approx 38.7^\circ$$

Now we can find A two different ways.

Method 1: We can using trigonometry and the cosine ratio:

$$\cos A = \frac{5}{8}$$
$$m\angle A = \cos^{-1}\left(\frac{5}{8}\right) \approx 51.3^\circ$$

Method 2: We can subtract $m\angle B$ from 90°: $90^\circ - 38.7^\circ = 51.3^\circ$ since the acute angles in a right triangle are always complimentary.

Either method is valid, but be careful with Method 2 because a miscalculation of angle B would make the measure you get for angle A incorrect as well.

Guided Practice

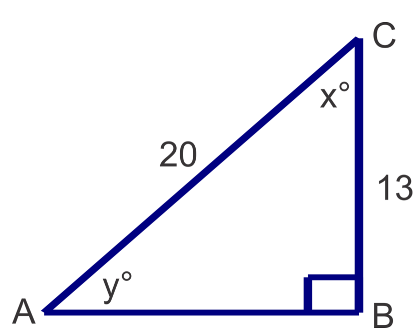
1. Find the measure of angle A to the nearest degree given the trigonometric ratios.

a. $\sin A = 0.2894$

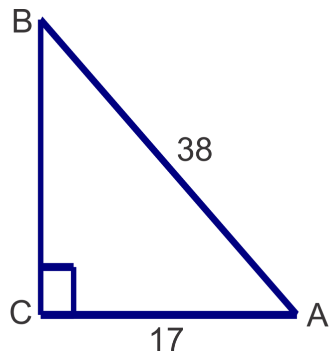
b. $\tan A = 2.1432$

c. $\cos A = 0.8911$

2. Find the measures of the unknown angles in the triangle shown. Round answers to the nearest degree.



3. Solve the triangle. Round side lengths to the nearest tenth and angles to the nearest degree.



Answers

1. a. $\sin^{-1}(0.2894) \approx 17°$

b. $\tan^{-1}(2.1432) \approx 65°$

c. $\cos^{-1}(0.8911) \approx 27°$

2.

$$x = \cos^{-1}\left(\frac{13}{20}\right) \approx 49°; \quad y = \sin^{-1}\left(\frac{13}{20}\right) \approx 41°$$

3.

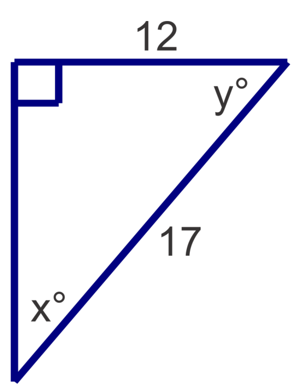
$$m\angle A = \cos^{-1}\left(\frac{17}{38}\right) \approx 63°; \quad m\angle B = \sin^{-1}\left(\frac{17}{38}\right) \approx 27°; \quad a = \sqrt{38^2 - 17^2} \approx 34.0$$

Problem Set

Use your calculator to find the measure of angle B. Round answers to the nearest degree.

1. $\tan B = 0.9523$
2. $\sin B = 0.8659$
3. $\cos B = 0.1568$

Find the measures of the unknown acute angles. Round measures to the nearest degree.



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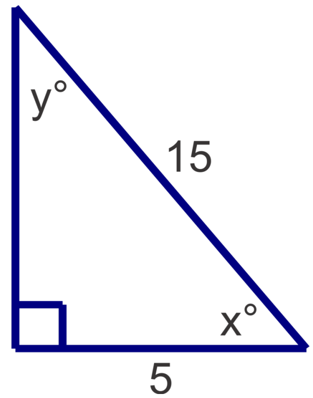

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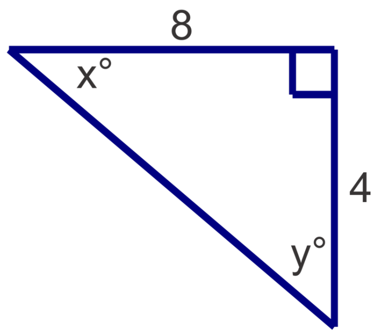

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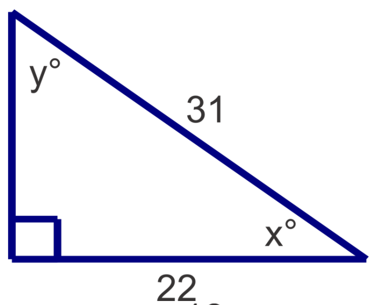

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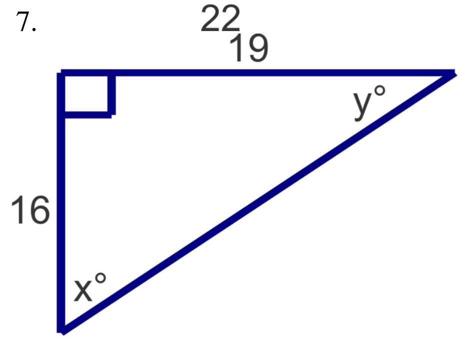

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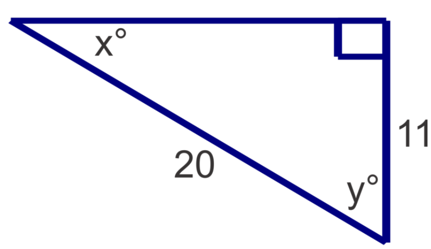
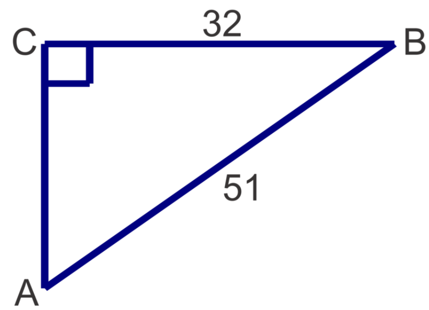
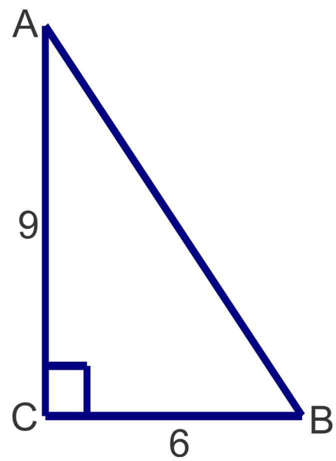
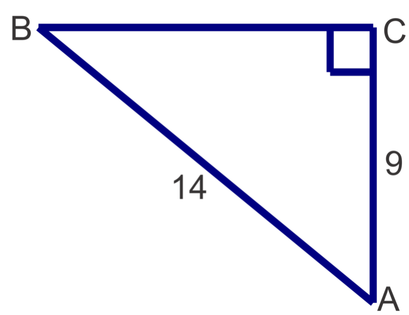


9.

Solve the following right triangles. Round angle measures to the nearest degree and side lengths to the nearest tenth.



10. 

11. 

12.

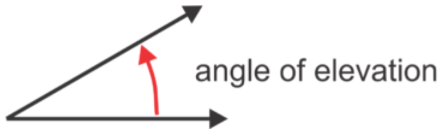
Application Problems

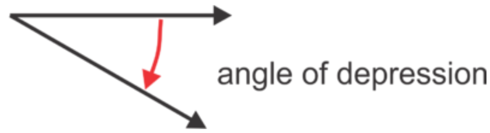
Objective

Use the Pythagorean Theorem and trigonometric ratios to solve the real world application problems.

Guidance

When solving word problems, it is important to understand the terminology used to describe angles. In trigonometric problems, the terms angle of elevation and angle of depression are commonly used. Both of these angles are always measured from a horizontal line as shown in the diagrams below.

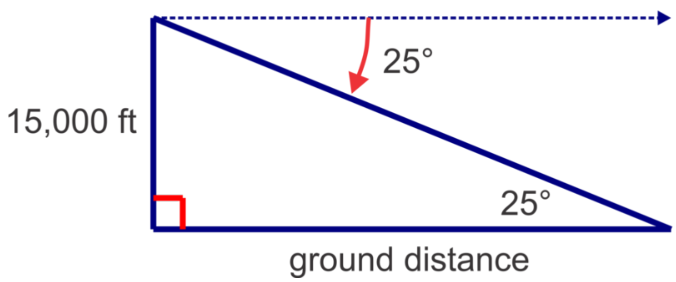




Example A

An airplane approaching an airport spots the runway at an angle of depression of 25°. If the airplane is 15,000 ft above the ground, how far (ground distance) is the plane from the runway? Give your answer to the nearest 100 ft.

Solution: Make a diagram to illustrate the situation described and then use a trigonometric ratio to solve. Keep in mind that an angle of depression is down from a horizontal line of sight-in this case a horizontal line from the pilot of the plane parallel to the ground.



Note that the angle of depression and the alternate interior angle will be congruent, so the angle in the triangle is also 25°.

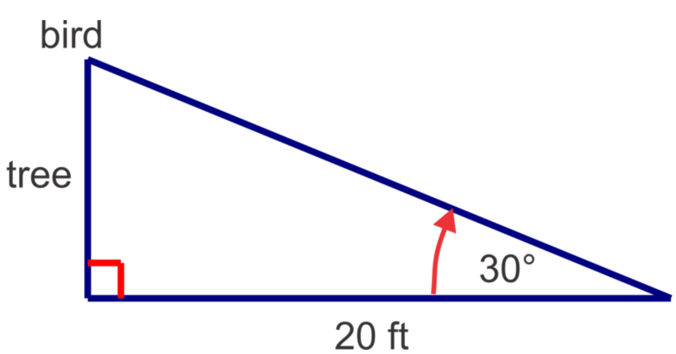
From the picture, we can see that we should use the tangent ratio to find the ground distance.

$$\begin{aligned}
\tan 25^\circ &= \frac{15000}{d}\\
d &= \frac{15000}{\tan 25^\circ} \approx 32,200\ ft
\end{aligned}$$

Example B

Rachel spots a bird in a tree at an angle of elevation of 30°. If Rachel is 20 ft from the base of the tree, how high up in the tree is the bird? Give your answer to the nearest tenth of a foot.

Solution: Make a diagram to illustrate the situation. Keep in mind that there will be a right triangle and that the right angle is formed by the ground and the trunk of the tree.



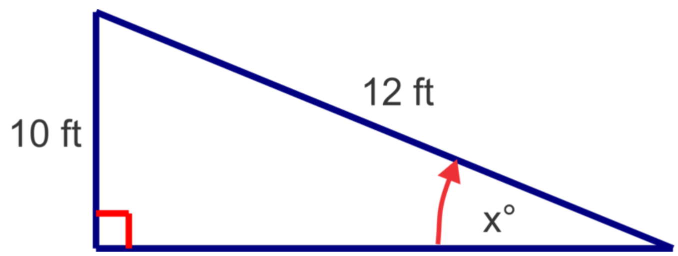
Here we can use the tangent ratio to solve for the height of the bird

$$\tan 30^\circ = \frac{h}{20}$$
$$h = 20\tan 30^\circ \approx 11.5\ ft$$

Example C

A 12 ft ladder is leaning against a house and reaches 10 ft up the side of the house. To the nearest degree, what angle does the ladder make with the ground?

Solution: In this problem, we will need to find an angle. By making a sketch of the triangle we can see which inverse trigonometric ratio to use.



$$\sin x^\circ = \frac{10}{12}$$
$$\sin^{-1}\left(\frac{10}{12}\right) \approx 56^\circ$$

Guided Practice

Use a trigonometry to solve the following application problems.

1. A ramp makes a 20° angle with the ground. If door the ramp leads to is 2 ft above the ground, how long is the ramp? Give your answer to the nearest tenth of a foot.

2. Charlie lets out 90 ft of kite string. If the angle of elevation of the string is 70°, approximately how high is the kite? Give your answer to the nearest foot.

3. A ship's sonar spots a wreckage at an angle of depression of 32°. If the depth of the ocean is about 250 ft, how far is the wreckage (measured along the surface of the water) from the ship, to the nearest foot.

Answers

1.

$$\sin 20^\circ = \frac{2}{x}$$
$$x = \frac{2}{\sin 20^\circ} \approx 5.8\ ft$$

2.

$$\sin 70^\circ = \frac{x}{90}$$
$$x = 90\sin 70^\circ \approx 85\ ft$$

3.

$$\tan 32^\circ = \frac{250}{x}$$
$$x = \frac{250}{\tan 32^\circ} \approx 400\ ft$$

Vocabulary

Angle of Elevation
An angle measured up from a horizontal line.

Angle of Depression
An angle measured down from a horizontal line.

Problem Set

Use the Pythagorean Theorem and/or trigonometry to solve the following word problems.

1. A square has sides of length 8 inches. To the nearest tenth of an inch, what is the length of its diagonal?
2. Layne spots a sailboat from her fifth floor balcony, about 25 m above the beach, at an angle of depression of 3°. To the nearest meter, how far out is the boat?
3. A zip line takes passengers on a 200 m ride from high up in the trees to a ground level platform. If the angle of elevation of the zip line is 10°, how high above ground is the tree top start platform? Give your answer to the nearest meter.
4. The angle of depression from the top of an apartment building to the base of a fountain in a nearby park is 57°. If the building is 150 ft tall, how far away, to the nearest foot, is the fountain?
5. A playground slide platform is 6 ft above ground. If the slide is 8 ft long and the end of the slide is 1 ft above ground, what angle does the slide make with the ground? Give your answer to the nearest degree.
6. Benjamin spots a tree directly across the river from where he is standing. He then walks 27 ft upstream and determines that the angle between his previous position and the tree on the other side of the river is 73°. How wide, to the nearest foot, is the river?
7. A rectangle has sides of length 6 in and 10 in. To the nearest degree, what angle does the diagonal make with the longer side?
8. Tommy is flying his kite one afternoon and notices that he has let out the entire 130 ft of string. The angle his string makes with the ground is 48°. How high, to the nearest foot, is his kite at this time?
9. A tree struck by lightning in a storm breaks and falls over to form a triangle with the ground. The tip of the tree makes a 18° angle with the ground 21 ft from the base of the tree. What was the height of the tree to the nearest foot?
10. Upon descent an airplane is 19,000 ft above the ground. The air traffic control tower is 190 ft tall. It is determined that the angle of elevation from the top of the tower to the plane is 15°. To the nearest mile, find the ground distance from the airplane to the tower.
11. Why will the sine and cosine ratios always be less than 1?

1.9 The Law of Sines

Objective

Use the Law of Sines proportion to solve non right triangles and find the area of triangles.

Review Queue

Evaluate the following trig functions. Give exact answers.

1. $\sin 225^\circ$
2. $\csc 300^\circ$
3. $\sec \frac{5\pi}{6}$
4. $\tan \pi$

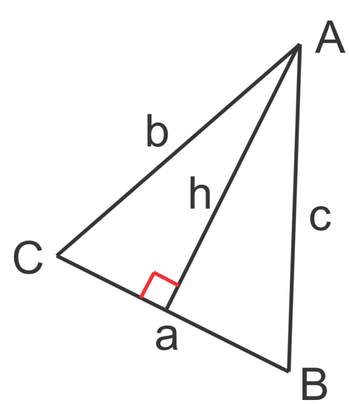
Law of Sines with AAS and ASA

Objective

Derive the Law of Sines proportion and use it to solve non right triangles in which two angles and one side are given.

Guidance

Consider the non right triangle below. We can construct an altitude from any one of the vertices to divide the triangle into two right triangles as show below.



from the diagram we can write two trigonometric functions involving h:

$$\begin{aligned} \sin C &= \frac{h}{b} \qquad \textit{and} \qquad & \sin B &= \frac{h}{c} \\ b\sin C &= h & c\sin B &= h \end{aligned}$$

Since both are equal to h, we can set them equal to each other to get:

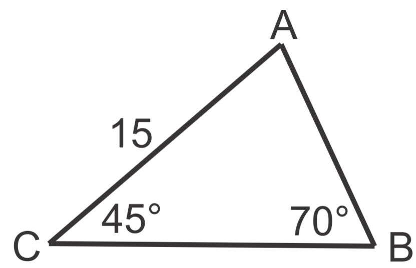
$b\sin C = c\sin B$ and finally divide both sides by bc to create the proportion:

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

If we construct the altitude from a different vertex, say B, we would get the proportion: $\frac{\sin A}{a} = \frac{\sin C}{c}$. Now, the transitive property allows us to conclude that $\frac{\sin A}{a} = \frac{\sin B}{b}$. We can put them all together as the Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$. In the examples that follow we will use the Law of Sines to solve triangles.

Example A

Solve the triangle.



Solution: Since we are given two of the three angles in the triangle, we can find the third angle using the fact that the three angles must add up to 180°. So, $m\angle A = 180^\circ - 45^\circ - 70^\circ = 650^\circ$. Now we can substitute the known values into the Law of Sines proportion as shown:

$$\frac{\sin 65^\circ}{a} = \frac{\sin 70^\circ}{15} = \frac{\sin 45^\circ}{c}$$

Taking two ratios at a time we can solve the proportions to find a and c using cross multiplication.

To find a:

$$\frac{\sin 65^\circ}{a} = \frac{\sin 70^\circ}{15}$$
$$a = \frac{15 \sin 65^\circ}{\sin 70^\circ} \approx 14.5$$

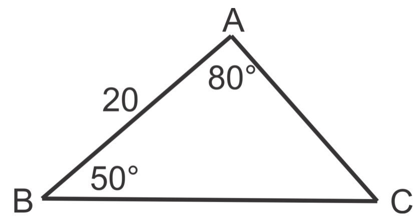
To find c:

$$\frac{\sin 70^\circ}{15} = \frac{\sin 45^\circ}{c}$$
$$c = \frac{15 \sin 45^\circ}{\sin 70^\circ} \approx 11.3$$

This particular triangle is an example in which we are given two angles and the non-included side or AAS (also SAA).

Example B

Solve the triangle.



Solution: In this example we are given two angles and a side as well but the side is between the angles. We refer to this arrangement as ASA. In practice, in doesn't really matter whether we are given AAS or ASA. We will follow the same procedure as Example A. First, find the third angle: $m\angle A = 180^\circ - 50^\circ - 80^\circ = 50^\circ$.

Second, write out the appropriate proportions to solve for the unknown sides, a and b.

To find a:

$$\frac{\sin 80^\circ}{a} = \frac{\sin 50^\circ}{20}$$
$$a = \frac{20 \sin 80^\circ}{\sin 50^\circ} \approx 25.7$$

To find b:

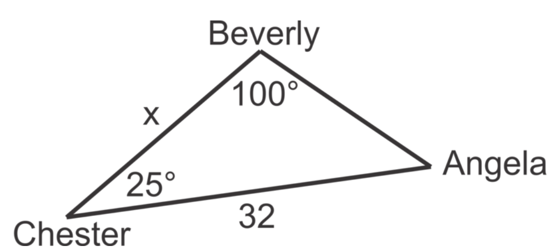
$$\frac{\sin 50^\circ}{b} = \frac{\sin 50^\circ}{20}$$
$$b = \frac{20 \sin 50^\circ}{\sin 50^\circ} = 20$$

Notice that $c = b$ and $m\angle C = m\angle B$. This illustrates a property of isosceles triangles that states that the base angles (the angles opposite the congruent sides) are also congruent.

Example C

Three fishing ships in a fleet are out on the ocean. The Chester is 32 km from the Angela. An officer on the Chester measures the angle between the Angela and the Beverly to be 25°. An officer on the Beverly measures the angle between the Angela and the Chester to be 100°. How far apart, to the nearest kilometer are the Chester and the Beverly?

Solution: First, draw a picture. Keep in mind that when we say that an officer on one of the ships is measuring an angle, the angle she is measuring is at the vertex where her ship is located.



Now that we have a picture, we can determine the angle at the Angela and then use the Law of Sines to find the distance between the Beverly and the Chester.

The angle at the Angela is $180^\circ - 100^\circ - 25^\circ = 55^\circ$.

Now find x,

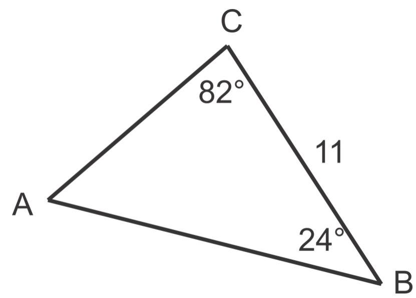
$$\begin{aligned}\frac{\sin 55^\circ}{x} &= \frac{\sin 100^\circ}{32} \\ x &= \frac{32\sin 55^\circ}{\sin 100^\circ} \approx 27\end{aligned}$$

The Beverly and the Chester are about 27 km apart.

Guided Practice

Solve the triangles.

1.



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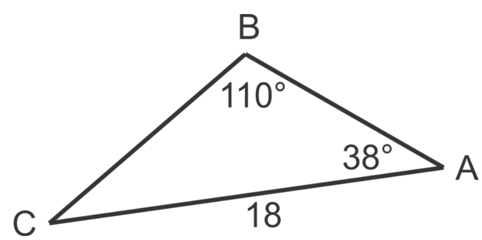


3. A surveying team is measuring the distance between point A on one side of a river and point B on the far side of the river. One surveyor is positioned at point A and the second surveyor is positioned at point C, 65 m up the riverbank from point A. The surveyor at point A measures the angle between points B and C to be 103°. The surveyor at point C measures the angle between points A and B to be 42°. Find the distance between points A and B.

Answers

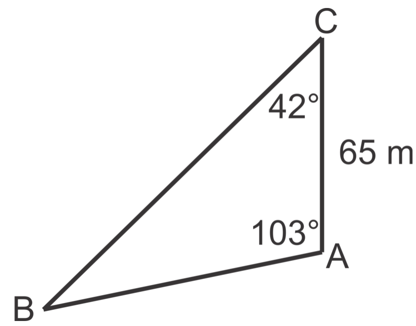
1. $m\angle A = 180^\circ - 82^\circ - 24^\circ = 74^\circ$

$$\begin{aligned}\frac{\sin 24^\circ}{b} &= \frac{\sin 74^\circ}{11}, \text{ so } b = \frac{11\sin 24^\circ}{\sin 74^\circ} \approx 4.7 \\ \frac{\sin 82^\circ}{c} &= \frac{\sin 74^\circ}{11}, \text{ so } c = \frac{11\sin 82^\circ}{\sin 74^\circ} \approx 11.3\end{aligned}$$

2. $m\angle C = 180^\circ - 110^\circ - 38^\circ = 32^\circ$

$$\begin{aligned}\frac{\sin 38^\circ}{a} &= \frac{\sin 110^\circ}{18}, \text{ so } a = \frac{18\sin 38^\circ}{\sin 110^\circ} \approx 11.8 \\ \frac{\sin 32^\circ}{c} &= \frac{\sin 110^\circ}{18}, \text{ so } c = \frac{18\sin 32^\circ}{\sin 110^\circ} \approx 10.2\end{aligned}$$

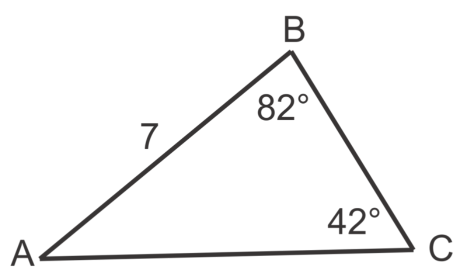
3.

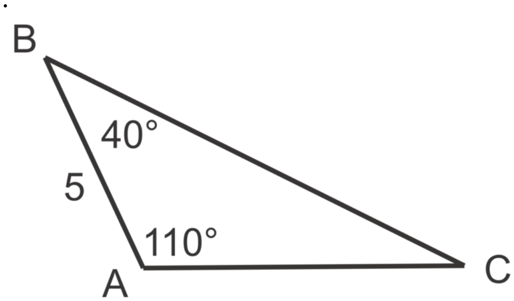


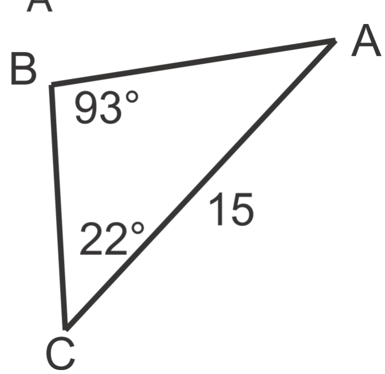
$$m\angle B = 180^\circ - 103^\circ - 42^\circ = 35^\circ$$
$$\frac{\sin 35^\circ}{65} = \frac{\sin 42^\circ}{c}$$
$$c = \frac{65 \sin 42^\circ}{\sin 35^\circ} \approx 75.8\ m$$

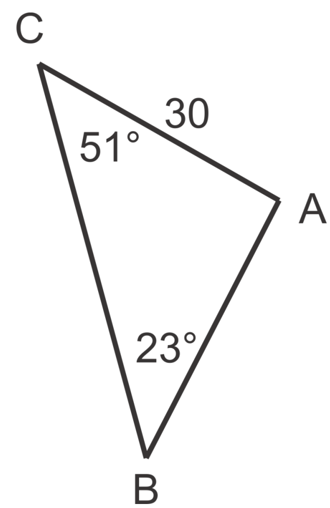
Problem Set

Solve the triangles. Round your answers to the nearest tenth.

1. 

2. 

3. 



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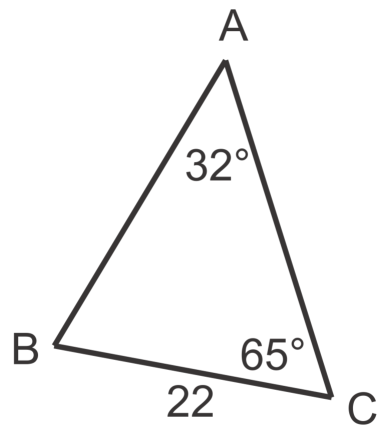

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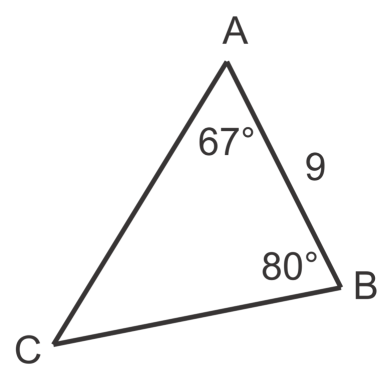


6.

Using the given information, solve ΔABC.

7.

$$m\angle A = 85^\circ$$
$$m\angle C = 40^\circ$$
$$a = 12$$

8.

$$\begin{aligned} m\angle B &= 60^\circ \\ m\angle C &= 25^\circ \\ a &= 28 \end{aligned}$$

9.

$$\begin{aligned} m\angle B &= 42^\circ \\ m\angle A &= 36^\circ \\ b &= 8 \end{aligned}$$

10.

$$\begin{aligned} m\angle B &= 30^\circ \\ m\angle A &= 125^\circ \\ c &= 45 \end{aligned}$$

Use the Law of Sines to solve the following world problems.

11. A surveyor is trying to find the distance across a ravine. He measures the angle between a spot on the far side of the ravine, X, and a spot 200 ft away on his side of the ravine, Y, to be 100°. He then walks to Y the angle between X and his previous location to be 20°. How wide is the ravine?
12. A triangular plot of land has angles 46° and 58°. The side opposite the 46° angle is 35 m long. How much fencing, to the nearest half meter, is required to enclose the entire plot of land?

The Ambiguous Case - SSA

Objective

When given two sides and the non-included angle, identify triangles in which there could be two solutions and find both if applicable.

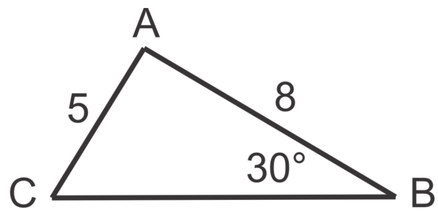
Guidance

Recall that the sine ratios for an angle and its supplement will always be equal. In other words, $\sin\theta = \sin(180-\theta)$. In Geometry you learned that two triangles could not be proven congruent using SSA and you investigated cases in which there could be two triangles. In Example A, we will explore how the Law of Sines can be used to find two possible triangles when given two side lengths of a triangle and a non-included angle.

Example A

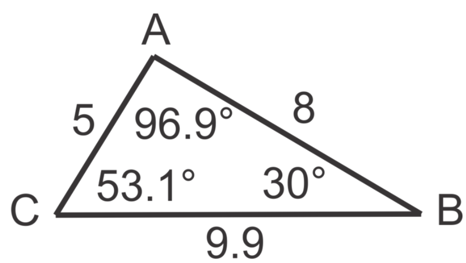
Given ΔABC with $m\angle A = 30^\circ$, $a = 5$, and $b = 8$, solve for the other angle and side measures.

Solution: First, let's make a diagram to show the relationship between the given sides and angles. Then we can set up a proportion to solve for angle C:



$$\begin{aligned}\frac{\sin 30^\circ}{5} &= \frac{\sin C}{8}\\ \sin C &= \frac{8\sin 30^\circ}{5}\\ C &= \sin^{-1}\left(\frac{8\sin 30^\circ}{5}\right) \approx 53.1^\circ\end{aligned}$$

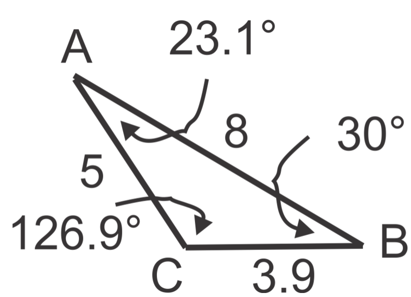
From here we can find $m\angle A = 96.9^\circ$, since the three angles must add up to 180°. We can also find the third side using another Law of Sines ratio:



$$\begin{aligned}\frac{\sin 30^\circ}{5} &= \frac{\sin 96.9^\circ}{a}\\ a &= \frac{5\sin 96.9^\circ}{\sin 30^\circ} \approx 9.9\end{aligned}$$

Putting these measures in the triangle, we get:

But, we know that $\sin\theta = \sin(180-\theta)$ so when we solved for C we only got one of the two possible angles. The other angle will be $180^\circ - 53.1^\circ = 126.9^\circ$. Next we need to determine the measure of angle A for and the length of the third side in this second possible triangle. The sum of the three angles must still be 180°, so $m\angle A = 23.1^\circ$. Now set up a proportion to solve for the third side just as before:



$$\begin{aligned}\frac{\sin 30^\circ}{5} &= \frac{\sin 23.1^\circ}{a}\\ a &= \frac{5\sin 23.1^\circ}{\sin 30^\circ} \approx 3.9\end{aligned}$$

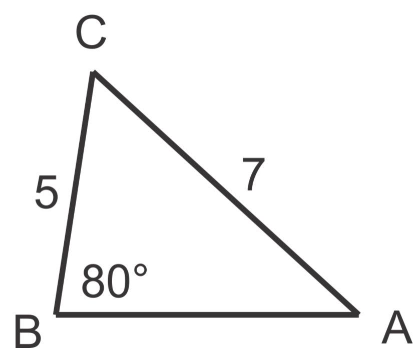
The second triangle would look like this:

In this instance there were two possible triangles.

Example B

Given ΔABC with $m\angle B = 80^\circ$, $a = 5$ and $b = 7$, solve for the other angle and side measures.

Solution: Again we will start with a diagram and use the law of sines proportion to find a second angle measure in the triangle.



$$\begin{aligned}\frac{\sin 80^\circ}{7} &= \frac{\sin A}{5}\\ \sin A &= \frac{5\sin 80^\circ}{7}\\ A &= \sin^{-1}\left(\frac{5\sin 80^\circ}{7}\right) \approx 44.7^\circ\end{aligned}$$

Now find the third angle, $180^\circ - 80^\circ - 44.7^\circ = 55.3^\circ$ and solve for the third side:

$$\begin{aligned}\frac{\sin 80^\circ}{7} &= \frac{\sin 55.3^\circ}{c}\\ c &= \frac{7\sin 55.3^\circ}{\sin 80^\circ} \approx 5.8\end{aligned}$$

Because we used the inverse sine function to determine the measure of angle A, the angle could be the supplement of 44.7° or 135.3° so we need to check for a second triangle. If we let $m\angle A = 135.3^\circ$ and then attempt to find the third angle, we will find that the sum of the two angles we have is greater than 180° and thus no triangle can be formed.

$$\begin{aligned}m\angle A + m\angle B + m\angle C &= 180^\circ\\ 135.3^\circ + 80^\circ + m\angle C &= 180^\circ\\ 215.3^\circ + m\angle C &> 180^\circ\end{aligned}$$

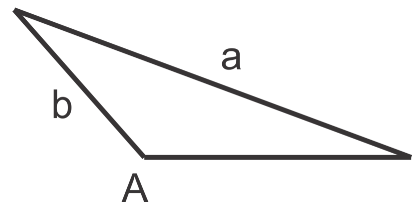
This example shows that two triangles are not always possible. Note that if the given angle is obtuse, there will only be one possible triangle for this reason.

More Guidance

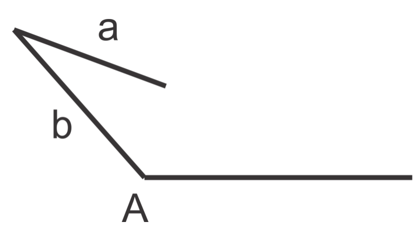
In both examples we simply tested to see if there would be a second triangle. There are, however, guidelines to follow to determine when a second triangle exists and when it does not. The "check and see" method always works and therefore it is not necessary to memorize the following table. It is interesting, however, to see to pictures and make the connection between the inequalities and what if any triangle can be formed.

First, consider when A is obtuse:

If $a > b$, then **one** triangle can be formed.

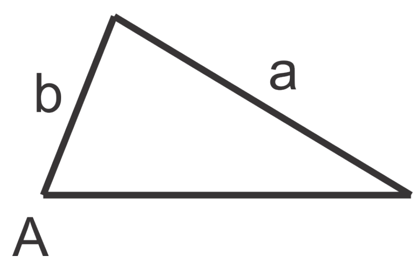


If $a \leq b$, then **no** triangle can be formed.



Now, consider the possible scenarios when A is acute.

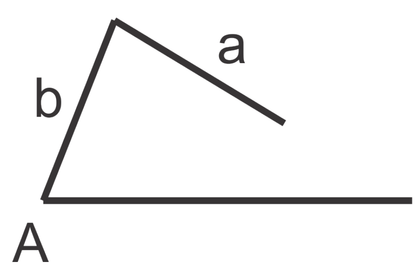
If $a > b$, the **one** triangle can be formed.



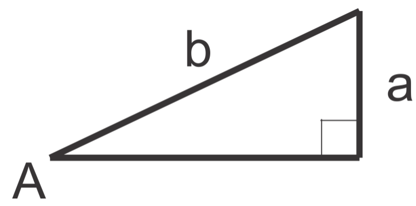
For the following cases, where $a < b$, keep in mind that we would be using the proportion:

$\frac{\sin A}{a} = \frac{\sin B}{b}$ and that $\sin B = \frac{b \sin A}{a}$

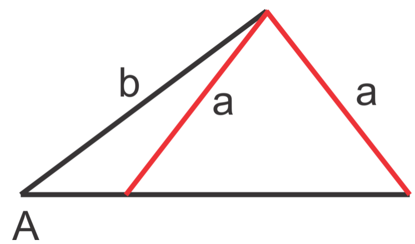
If $b \sin A > a$, **no** triangle can be formed because $B > 1$.



If $b \sin A = a$, **one right** triangle can be formed because $\sin B = 1$.



If $b\sin A < a$ (and $a < b$), **two** triangles can be formed because $\sin B < 1$.



Example C

Given ΔABC with $m\angle A = 42^\circ$, $b = 10$ and $a = 8$, use the rules to determine how many, if any, triangles can be formed and then solve the possible triangle(s).

Solution: In this case, A is acute and $a < b$, so we need to look at the value of $b\sin a$. Since $b\sin A = 10\sin 42^\circ \approx 6.69 < a$, there will be two triangles. To solve for these triangles, use the Law of Sines extended proportion instead of making a diagram. Plugging in what we know, we have:

$$\frac{\sin 42^\circ}{8} = \frac{\sin B}{b} = \frac{\sin C}{10}$$

Take the first and last ratios to solve a proportion to find the measure of angle A.

$$\frac{\sin C}{10} = \frac{\sin 42^\circ}{8}$$
$$C = \sin^{-1}\left(\frac{10\sin 42^\circ}{8}\right) \approx 56.8^\circ$$

So, the $m\angle C \approx 56.8^\circ$ or 123.2° and $m\angle B \approx 81.2^\circ$ or 14.8° respectively.

Solve for the measure of side b in each triangle:

$$\frac{\sin 42^\circ}{8} = \frac{\sin 81.2^\circ}{b} \qquad \textit{and} \qquad \frac{\sin 42^\circ}{8} = \frac{\sin 14.8^\circ}{b}$$
$$b = \frac{8\sin 81.2^\circ}{\sin 42^\circ} \approx 11.8 \qquad\qquad b = \frac{8\sin 14.8^\circ}{\sin 42^\circ} \approx 3.1$$

Putting it all together, we have:

Triangle 1: $m\angle A \approx 42^\circ, m\angle B \approx 81.2^\circ, m\angle C = 56.8^\circ, a = 8, b \approx 11.8, c = 10$

Triangle 2: $m\angle A \approx 42^\circ, m\angle B \approx 14.8^\circ, m\angle C = 123.2^\circ, a = 8, b \approx 3.1, c = 10$

Guided Practice

1. Use the given side lengths and angle measure to determine whether zero, one or two triangles exists.

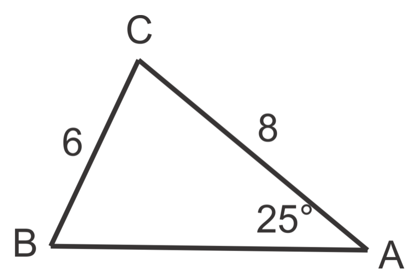
a. $m\angle A = 100^\circ, a = 3, b = 4$.

b. $m\angle A = 50^\circ, a = 8, b = 10$.

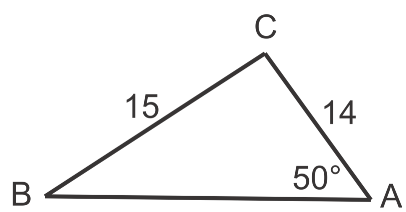
c. $m\angle A = 72^\circ, a = 7, b = 6$.

2. Solve the following triangles.

a.



b.



3. Given $m\angle A = 30^\circ$, $a = 80$ and $b = 150$, find $m\angle C$.

Answers

1. a. Since A is obtuse and $a \le b$, no triangle can be formed.

b. Since A is acute, $a < b$ and $b \sin A < a$, two triangles can be formed.

c. Since A is acute and $a > b$, there is one possible triangle.

2. a. There will be two triangles in this case because A is acute, $a < b$ and $b \sin A < a$.

Using the extended proportion: $\frac{\sin 25^\circ}{6} = \frac{\sin B}{8} = \frac{\sin C}{c}$, we get:

$$\begin{aligned} m\angle B &\approx 34.3^\circ & \quad or \quad & & m\angle B &\approx 145.7^\circ \\ m\angle C &\approx 120.7^\circ & & & m\angle C &\approx 9.3^\circ \\ c &\approx 12.2 & & & c &\approx 2.3 \end{aligned}$$

b. Since A is acute and $a > b$, there is one possible triangle.

Using the extended proportion: $\frac{\sin 50^\circ}{15} = \frac{\sin B}{14} = \frac{\sin C}{c}$, we get:

$$\begin{aligned} m\angle B &\approx 45.6^\circ \\ m\angle C &\approx 84.4^\circ \\ c &\approx 19.5 \end{aligned}$$

3. In this instance A is acute, $a < b$ and $b \sin A < a$ so two triangles can be formed. So, once we find the two possible measures of angle B, we will find the two possible measures of angle C. First find $m\angle B$:

$$\frac{\sin 30^\circ}{80} = \frac{\sin B}{150}$$
$$\sin B = \frac{150 \sin 30^\circ}{80}$$
$$B \approx 69.6^\circ, 110.4^\circ$$

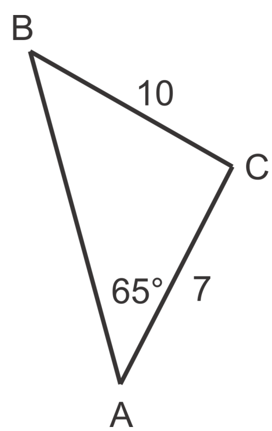
Now that we have B, use the triangle sum to find $m\angle C \approx 80.4^\circ, 39.9^\circ$.

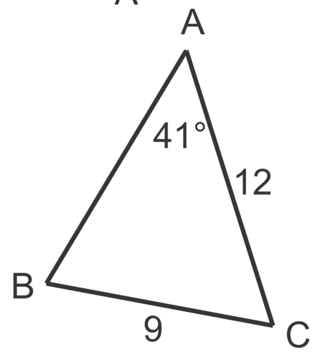
Problem Set

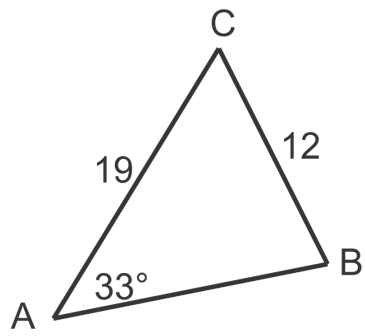
For problems 1-5, use the rules to determine if there will be one, two or no possible triangle with the given measurements.

1. $m\angle A = 65^\circ, a = 10, b = 11$
2. $m\angle A = 25^\circ, a = 8, b = 15$
3. $m\angle A = 100^\circ, a = 6, b = 4$
4. $m\angle A = 75^\circ, a = 25, b = 30$
5. $m\angle A = 48^\circ, a = 41, b = 50$

Solve the following triangles, if possible. If there is a second possible triangle, solve it as well.

6. 

7. 



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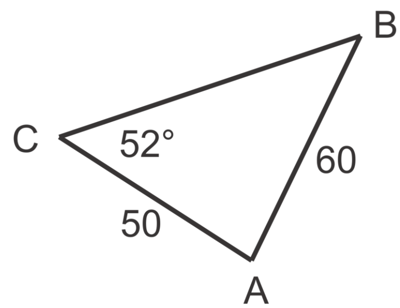

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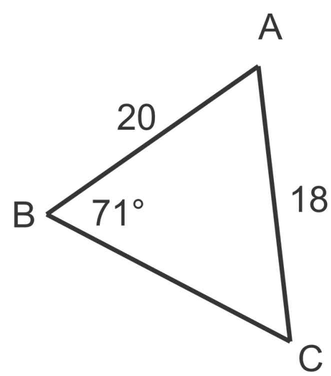

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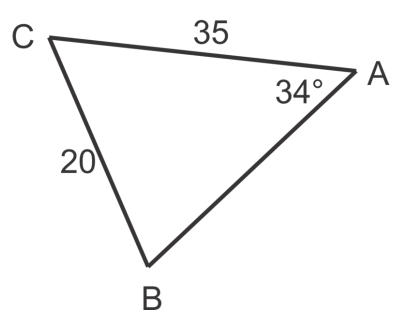


11.

Area of a Triangle

Objective

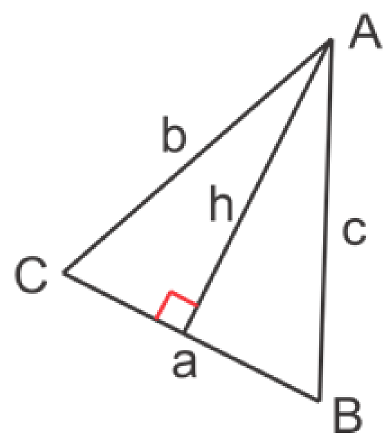
Use the sine ratio to find the area of non-right triangles in which two sides and the included angle measure are known.

Guidance

Recall the non right triangle for which we derived the law of sine.

We are most familiar with the area formula: $A = \frac{1}{2}bh$ where the base, b, is the side length which is perpendicular to the altitude. If we consider angle C in the diagram, we can write the following trigonometric expression for the altitude of the triangle, h:

$$\sin C = \frac{h}{b}$$
$$b \sin C = h$$



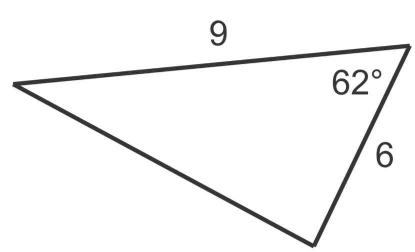
No we can replace h in the formula with $b \sin C$ and the side perpendicular to h is the base, a. Our new area formula is thus:

$$A = \frac{1}{2}ab \ \sin C.$$

It is important to note that C is the angle between sides a and b and that any two sides and the included angle can be used in the formula.

Example A

Find the area of the triangle.

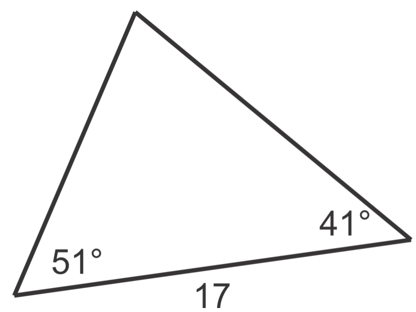


Solution: We are given two sides and the included angle so let $a = 6$, $b = 9$ and $C = 62^\circ$. Now we can use the formula to find the area of the triangle:

$$A = \frac{1}{2}(6)(9)\sin(62^\circ) \approx 23.8 \textit{ square units}$$

Example B

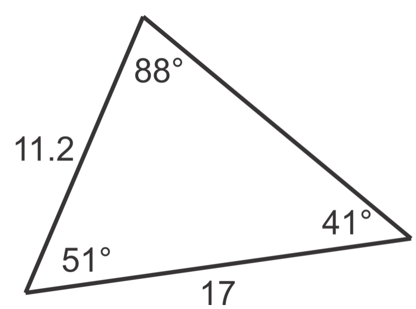
Find the area of the triangle.



Solution: In this triangle we do not have two sides and the included angle. We must first find another side length using the Law of Sines. We can find the third angle using the triangle sum: $180° - 51° - 41° = 88°$. Use the Law of Sines to find the side length opposite $41°$:

$$\frac{\sin 88°}{17} = \frac{\sin 41°}{x}$$
$$x = \frac{17 \sin 41°}{\sin 88°} \approx 11.2$$

Put these measures in the triangle:



We now have two sides and the included angle and can use the area formula:

$$A = \frac{1}{2}(11.2)(17)\sin(51°) \approx 74 \textit{ square units}$$

Example C

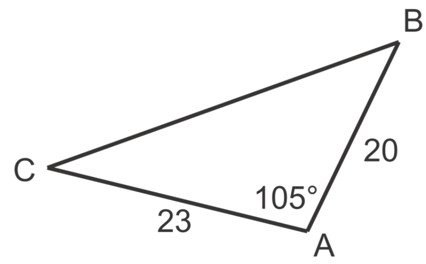
Given $c = 25\ cm$, $a = 31\ cm$ and $B = 78°$, find the area of ΔABC.

Solution: Here we are given two sides and the included angle. We can adjust the formula to represent the sides and angle we are given: $A = \frac{1}{2}ac\ \sin B$. It really doesn't matter which "letters" are in the formula as long as they represent **two sides and the included angle** (the angle *between* the two sides.) Now put in our values to find the area: $A = \frac{1}{2}(31)(25)\sin(78°) \approx 379\ cm^2$.

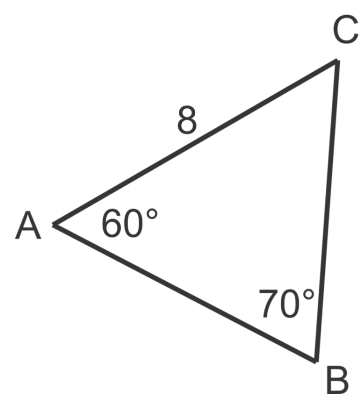
Guided Practice

Find the area of each of the triangles below. Round answers to the nearest square unit.

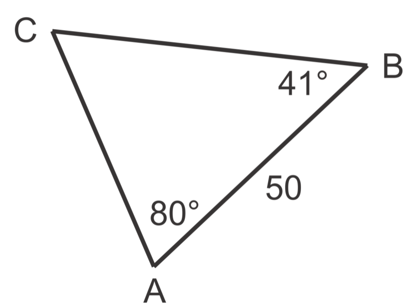
1.



2.



3.



Answers

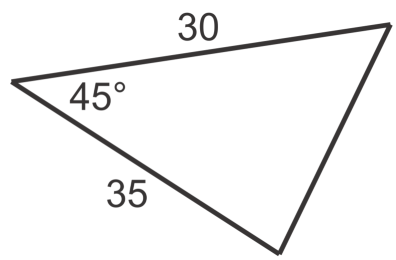
1. Two sides and the included angle are given so $A = \frac{1}{2}(20)(23)\sin 105° \approx 222\ sq\ units$.

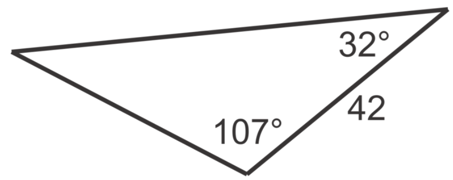
2. Find side a first: $\frac{\sin 70°}{8} = \frac{\sin 60°}{a}$, so $a = \frac{8 \sin 60°}{\sin 70°} \approx 7.4$. Next find $m\angle C = 180° - 60° - 70° = 50°$.
Using the area formula, $A = \frac{1}{2}(7.4)(8)\sin 50° \approx 22.7\ sq\ units$.

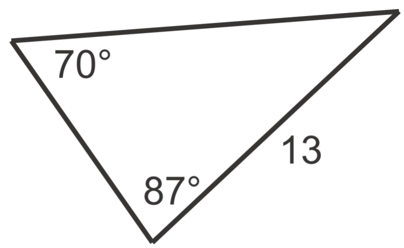
3. Find $m\angle C = 180° - 80° - 41° = 59°$. Find a second side: $\frac{\sin 59°}{50} = \frac{\sin 80°}{a}$, so $a = \frac{50 \sin 80°}{\sin 59°} \approx 57.4$.
Using the area formula, $A = \frac{1}{2}(57.4)(50)\sin 41° \approx 941\ sq\ units$.

Problem Set

Find the area of each of the triangles below. Round your answers to the nearest square unit.

1. 

2. 

3. 

4. $m\angle A = 71^\circ, b = 15, c = 19$
5. $m\angle C = 120^\circ, b = 22, a = 16$
6. $m\angle B = 60^\circ, a = 18, c = 12$
7. $m\angle A = 28^\circ, m\angle C = 73^\circ, b = 45$
8. $m\angle B = 56^\circ, m\angle C = 81^\circ, c = 33$
9. $m\angle A = 100^\circ, m\angle B = 30^\circ, a = 100$
10. The area of ΔABC is 66 square units. If two sides of the triangle are 11 and 21 units, what is the measure of the included angle? Is there more than one possible value? Explain.
11. A triangular garden is bounded on one side by a 20 ft long barn and a second side is bounded by a 25 ft long fence. If the barn and the fence meet at a 50° angle, what is the area of the garden if the third side is the length of the segment between the ends of the fence and the barn?
12. A contractor is constructing a counter top in the shape of an equilateral triangle with side lengths 3 ft. If the countertop material costs \$25 per square foot, how much will the countertop cost?

1.10 The Law of Cosines

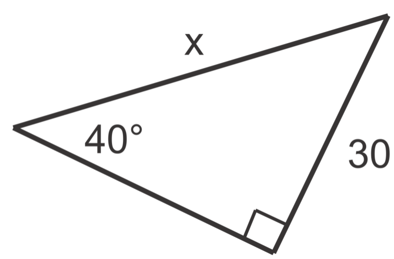
Objective

Use the Law of Cosines equation to solve non right triangles and find the area of triangles using Heron's Formula.

Review Queue

Find the value of x in the following triangles.

1.



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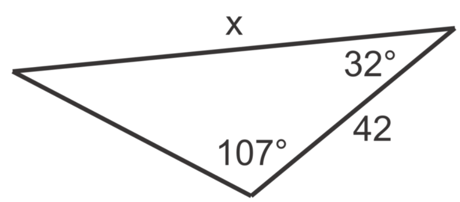

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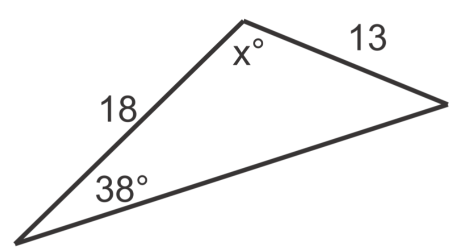


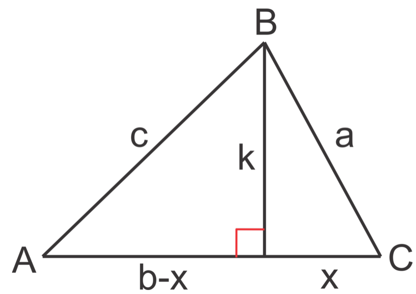
Using the Law of Cosines with SAS (to find the third side)

Objective

Use the Law of Cosines to determine the length of the third side of a triangle when two sides and the included angle are known.

Guidance

The Law of Cosines can be used to solve for the third side of a triangle when two sides and the included angle are known in a triangle. consider the non right triangle below in which we know a, b and C. We can draw an altitude from B to create two smaller right triangles as shown where x represents the length of the segment from C to the foot of the altitude and $b-x$ represents the length of remainder of the side opposite angle B.



Now we can use the Pythagorean Theorem to relate the lengths of the segments in each of the right triangles shown.

Triangle 1: $x^2+k^2=a^2$ or $k^2=a^2-x^2$

Triangle 2: $(b-x)^2+k^2=c^2$ or $k^2=c^2-(b-x)^2$

Since both equations are equal to k^2, we can set them equal to each other and simplify:

$$\begin{aligned}a^2-x^2&=c^2-(b-x)^2\\a^2-x^2&=c^2-(b^2-2bx+x^2)\\a^2-x^2&=c^2-b^2+2bx-x^2\\a^2&=c^2-b^2+2bx\\a^2+b^2-2bx&=c^2\end{aligned}$$

Recall that we know the values of a and b and the measure of angle C. We don't know the measure of x. We can use the cosine ratio as show below to find an expression for x in terms of what we already know.

$$\cos C=\frac{x}{a} \quad \text{so} \quad x=a\cos C$$

Finally, we can replace x in the equation to get the Law of Cosines: $a^2+b^2-2ab\cos C=c^2$

Keep in mind that a and b are the sides of angle C in the formula.

Example A

Find c when $m\angle C=80^\circ, a=6$ and $b=12$.

Solution: Replacing the variables in the formula with the given information and solve for c:

$$\begin{aligned}c^2&=6^2+12^2-2(6)(12)\cos 80^\circ\\c^2&\approx 154.995\\c&\approx 12.4\end{aligned}$$

Example B

Find a, when $m\angle A=43^\circ$, $b=16$ and $c=22$.

Solution: This time we are given the sides surrounding angle A and the measure of angle A. We can rewrite the formula as: $a^2=c^2+b^2-2cb\cos A$. Just remember that the length by itself on one side should be the side opposite the angle in the cosine ratio. Now we can plug in our values and solve for a.

$$
\begin{aligned}
a^2 &= 16^2 + 22^2 - 2(16)(22)\cos 43^\circ \\
a^2 &\approx 225.127 \\
a &\approx 15
\end{aligned}
$$

Example C

Rae is making a triangular flower garden. One side is bounded by her porch and a second side is bounded by her fence. She plans to put in a stone border on the third side. If the length of the porch is 10 ft and the length of the fence is 15 ft and they meet at a 100° angle, how many feet of stone border does she need to create?

Solution: Let the two known side lengths be a and b and the angle between is C. Now we can use the formula to find c, the length of the third side.

$$
\begin{aligned}
c^2 &= 10^2 + 15^2 - 2(10)(15)\cos 100^\circ \\
c^2 &\approx 377.094 \\
c &\approx 19.4
\end{aligned}
$$

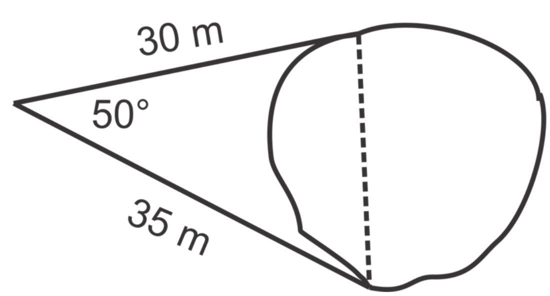
So Rae will need to create a 19.4 ft stone border.

Guided Practice

1. Find c when $m\angle C = 75^\circ, a = 32$ and $b = 40$.

2. Find b when $m\angle B = 120^\circ, a = 11$ and $c = 17$.

3. Dan likes to swim laps across a small lake near his home. He swims from a pier on the north side to a pier on the south side multiple times for a workout. One day he decided to determine the length of his swim. He determines the distances from each of the piers to a point on land and the angles between the piers from that point to be 50°. How many laps does Dan need to swim to cover 1000 meters?



Answers

1.

$$
\begin{aligned}
c^2 &= 32^2 + 40^2 - 2(32)(40)\cos 75^\circ \\
c^2 &\approx 1961.42 \\
c &\approx 44.3
\end{aligned}
$$

2.

$$
\begin{aligned}
b^2 &= 11^2 + 17^2 - 2(11)(17)\cos 120^\circ \\
b^2 &\approx 597 \\
b &\approx 24.4
\end{aligned}
$$

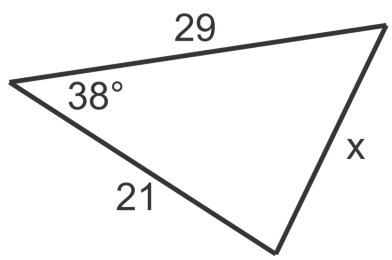
3.

$$
\begin{aligned}
c^2 &= 30^2 + 35^2 - 2(30)(35)\cos 50^\circ \\
c^2 &\approx 775.146 \\
c &\approx 27.84
\end{aligned}
$$

Since each lap is 27.84 meters, Dan must swim $\frac{1000}{27.84} \approx 36$ laps.

Problem Set

Use the Law of Cosines to find the value of x, to the nearest tenth, in problems 1 through 6.



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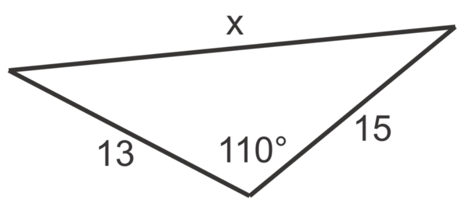

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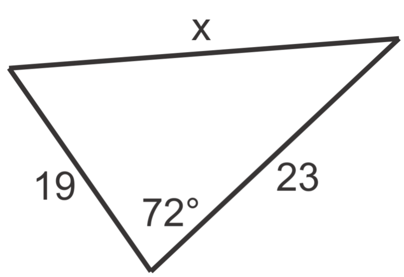

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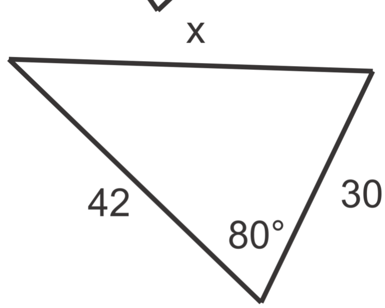

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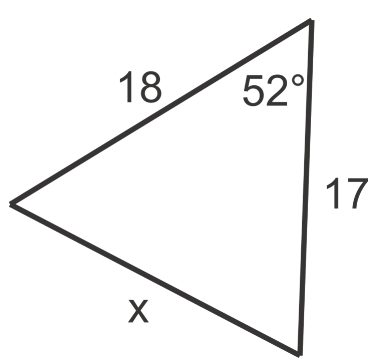

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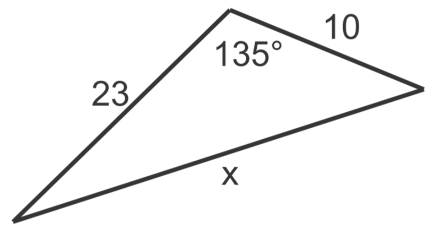


6.

For problems 7 through 10, find the unknown side of the triangle. Round your answers to the nearest tenth.

7. Find c, given $m\angle C = 105^\circ$, $a = 55$ and $b = 61$.
8. Find b, given $m\angle B = 26^\circ$, $a = 33$ and $c = 24$.
9. Find a, given $m\angle A = 77^\circ$, $b = 12$ and $c = 19$.
10. Find b, given $m\angle B = 95^\circ$, $a = 28$ and $c = 13$.
11. Explain why when $m\angle C = 90^\circ$, the Law of Cosines becomes the Pythagorean Theorem.
12. Luis is designing a triangular patio in his backyard. One side, 20 ft long, will be up against the side of his house. A second side is bordered by his wooden fence. If the fence and the house meet at a 120° angle and the fence is 15 ft long, how long is the third side of the patio?

Using the Law of Cosines with SSS (to find an angle)

Objective

Use the Law of Cosines to find the measure of an angle in a triangle in which all three side lengths are known.

Guidance

The Law of Cosines, $a^2 + b^2 - 2ab\cos C$, can be rearranged to facilitate the calculation of the measure of angle C when a, b and c are all known lengths.

$$\begin{aligned} a^2 + b^2 - 2ab\cos C &= c^2 \\ a^2 + b^2 - c^2 &= 2ab\cos C \\ \frac{a^2 + b^2 - c^2}{2ab} &= \cos C \end{aligned}$$

which can be further manipulated to $C = \cos^{-1}\left(\frac{a^2+b^2-c^2}{2ab}\right)$.

Example A

Find the measure of the largest angle in the triangle with side lengths 12, 18 and 21.

Solution: First, we must determine which angle will be the largest. Recall from Geometry that the longest side is opposite the largest angle. The longest side is 21 so we will let $c = 21$ since C is the angle we are trying to find. Let $a = 12$ and $b = 18$ and use the formula to solve for C as shown. It doesn't matter which sides we assign to a and b. They are interchangeable in the formula.

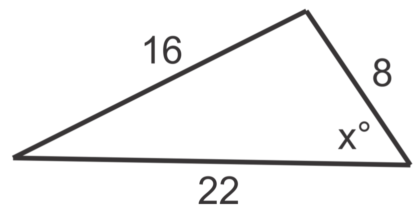
$$m\angle C = \cos^{-1}\left(\frac{12^2 + 18^2 - 21^2}{2(12)(18)}\right) \approx 86^\circ$$

Note: Be careful to put parenthesis around the entire numerator and entire denominator on the calculator to ensure the proper order of operations. Your calculator screen should look like this:

$$\cos^{-1}((12^2+18^2-21^2)/(2(12)(18)))$$

Example B

Find the value of x, to the nearest degree.



Solution: The angle with measure x° will be angle C so $c=16, a=22$ and $b=8$. Remember, a and b are interchangeable in the formula. Now we can replace the variables with the known measures and solve.

$$\cos^{-1}\left(\frac{22^2+8^2-16^2}{2(22)(8)}\right) \approx 34^\circ$$

Example C

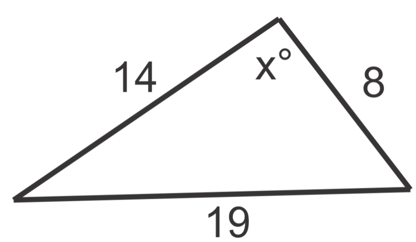
Find the $m\angle A$, if $a=10, b=15$ and $c=21$.

Solution: First, let's rearrange the formula to reflect the sides given and requested angle:

$\cos A = \left(\frac{b^2+c^2-a^2}{2(b)(c)}\right)$, now plug in our values $m\angle A = \cos^{-1}\left(\frac{15^2+21^2-10^2}{2(15)(21)}\right) \approx 26^\circ$

Guided Practice

1. Find the measure of x in the diagram:



2. Find the measure of the smallest angle in the triangle with side lengths 47, 54 and 72.

3. Find $m\angle B$, if $a=68, b=56$ and $c=25$.

Answers

1. $\cos^{-1}\left(\frac{14^2+8^2-19^2}{2(14)(8)}\right) \approx 117^\circ$

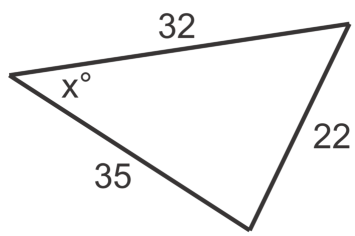
2. The smallest angle will be opposite the side with length 47, so this will be our c in the equation.

$$\cos^{-1}\left(\frac{54^2+72^2-47^2}{2(54)(72)}\right) \approx 41^\circ$$

3. Rearrange the formula to solve for $m\angle B, \cos B = \left(\frac{a^2+c^2-b^2}{2(a)(c)}\right); \cos^{-1}\left(\frac{68^2+25^2-56^2}{2(68)(25)}\right) \approx 52^\circ$

Problem Set

Use the Law of Cosines to find the value of x, to the nearest degree, in problems 1 through 6.



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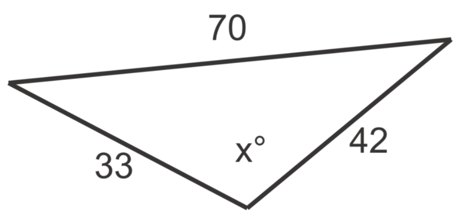

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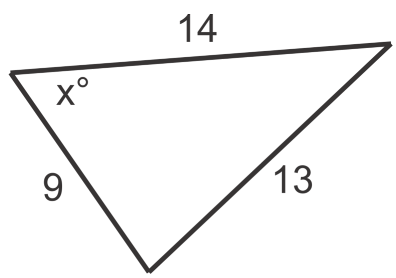

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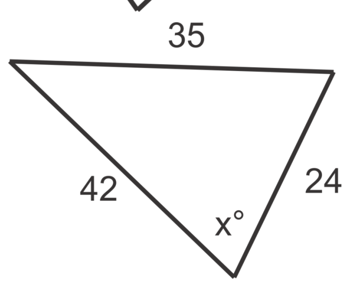

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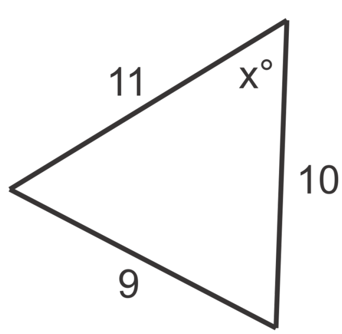

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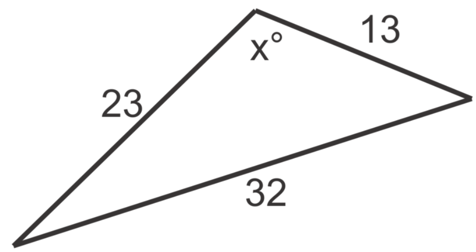


6.

7. Find the measure of the smallest angle in the triangle with side lengths 150, 165 and 200 meters.
8. Find the measure of the largest angle in the triangle with side length 59, 83 and 100 yards.
9. Find the $m\angle C$ if $a = 6, b = 9$ and $c = 13$.
10. Find the $m\angle B$ if $a = 15, b = 8$ and $c = 9$.
11. Find the $m\angle A$ if $a = 24, b = 20$ and $c = 14$.
12. A triangular plot of land is bordered by a road, a fence and a creek. If the stretch along the road is 100 meters, the length of the fence is 115 meters and the side along the creek is 90 meters, at what angle do the fence and road meet?

Heron's Formula for the Area of a Triangle and Problem Solving with Trigonometry

Objective

Use Heron's formula for area of a triangle when the side lengths are known and solve real world application problems using Law of Sines, Law of Cosines or the area formulas.

Guidance

Heron's Formula, named after Hero of Alexandria 2000 years ago, can be used to find the area of a triangle given the three side lengths. The formula requires the semi-perimeter, s, or $\frac{1}{2}(a+b+c)$, where a, b and c are the lengths of the sides of the triangle.

Heron's Formula:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

Example A

Use Heron's formula to find the area of a triangle with side lengths 13 cm, 16 cm and 23 cm.

Solution: First, find the semi-perimeter or s: $s = \frac{1}{2}(13+16+23) = 26$. Next, substitute our values into the formula as shown and evaluate:

$$A = \sqrt{26(26-13)(26-16)(26-23)} = \sqrt{26(13)(10)(3)} = \sqrt{10140} \approx 101\ cm^2$$

Example B

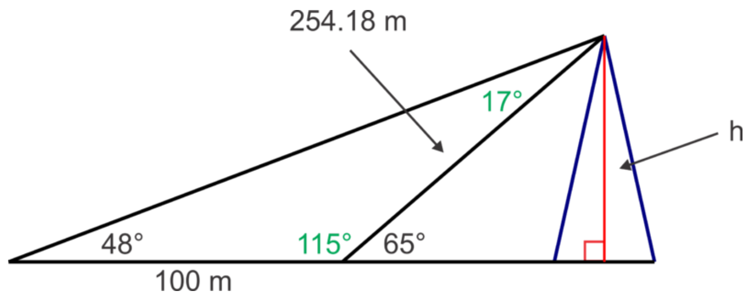
Alena is planning a garden in her yard. She is using three pieces of wood as a border. If the pieces of wood have lengths 4 ft, 6ft and 3 ft, what is the area of her garden?

Solution: The garden will be triangular with side lengths 4 ft, 6 ft and 3 ft. Find the semi-perimeter and then use Heron's formula to find the area.

$$s = \frac{1}{2}(4+6+3) = \frac{13}{2}$$
$$A = \sqrt{\frac{13}{2}\left(\frac{13}{2}-4\right)\left(\frac{13}{2}-6\right)\left(\frac{13}{2}-3\right)} = \sqrt{\frac{13}{2}\left(\frac{5}{2}\right)\left(\frac{1}{2}\right)\left(\frac{7}{2}\right)} = \sqrt{\frac{455}{16}} \approx 28\ ft^2$$

Example C

Caroline wants to measure the height of a radio tower. From some distance away from the tower, the angle of elevation from her spot to the top of the tower is 65°. Caroline walks 100 m further away from the tower and measures the angle of elevation to the top of tower to be 48°. How tall is the tower?



Solution: First, make a diagram to illustrate the situation.

We can use angle properties (linear pair and triangle sum) to find the angles shown in green in the diagram.

$180° - 65° = 115°$ and $180° - 48° - 115° = 17°$

Next, we can use the Law of Sines in the obtuse triangle to find the hypotenuse in the right triangle:

$$\frac{\sin 17°}{100} = \frac{\sin 48°}{x}$$
$$x = \frac{100 \sin 48°}{\sin 17°} \approx 254.18$$

Finally we can use the sine ratio in the right triangle to find the height of the tower:

$\sin 65° = \frac{h}{254.18}, h = 254.18 \sin 65° \approx 230.37\ m$

Guided Practice

Use the most appropriate rule or formula (Law of Sines, Law of Cosines, area formula with sine or Heron's formula) to answer the following questions.

1. Find the area of a triangle with side lengths 50 m, 45 m and 25 m.

2. Matthew is planning to fertilize his grass. Each bag of fertilizer claims to cover 500 sq ft of grass. His property of land is approximately in the shape of a triangle. He measures two sides of his yard to be 75 ft and 100 ft and the angle between them is 72°. How many bags of fertilizer must he buy?

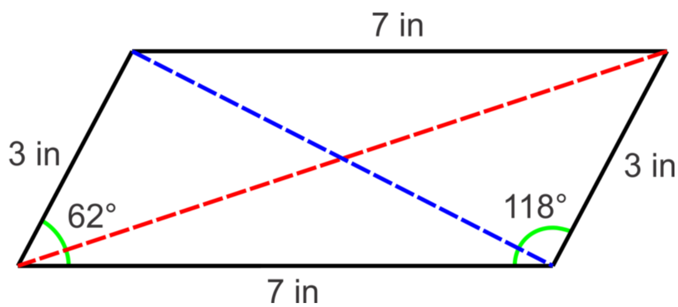
3. A pair of adjacent sides in a parallelogram are 3 in and 7 in and the angle between them is 62°, find the length of the diagonals.

Answers

1. Heron's Formula: $s = \frac{1}{2}(50+45+25) = 60, A = \sqrt{60(60-50)(60-45)(60-25)} \approx 561\ m^2$.

2. Area formula with sine: $\frac{1}{2}(75)(100)\sin 72° \approx 3566\ ft^2$, Number of bags $\frac{3566}{500} \approx 7.132 \approx 8$ bags. We round up because 7 bags is not quite enough.

3.



Law of Cosines to find the blue diagonal:

$$c^2 = 3^2 + 7^2 - 2(3)(7)\cos 62^\circ$$
$$c = \sqrt{38.28} \approx 6.19$$

So, 6.19 in

To find the green diagonal we can use the Law of Cosines with the adjacent angle: $180^\circ - 62^\circ - 118^\circ$:

$$c^2 = 7^2 + 3^2 - 2(7)(3)\cos 118^\circ$$
$$c = \sqrt{77.72} \approx 8.82$$

So, 8.82 in

Problem Set

Use the Law of Sines, Law of Cosine, area of triangle with sine or Heron's Formula to solve the real world application problems.

1. Two observers, Rachel and Luis, are standing on the shore, 0.5 miles apart. They each measure the angle between the shoreline and a sailboat out on the water at the same time. If Rachel's angle is 63° and Luis' angle is 56°, find the distance between Luis and the sailboat to the nearest hundredth of a mile.
2. Two pedestrians walk from opposite ends of a city block to a point on the other side of the street. The angle formed by their paths is 125°. One pedestrian walks 300 ft and the other walks 320 ft. How long is the city block to the nearest foot?
3. Two sides and the included angle of a parallelogram have measures 3.2 cm, 4.8 cm and 54.3° respectively. Find the lengths of the diagonals to the nearest tenth of a centimeter.
4. A bridge is supported by triangular braces. If the sides of each brace have lengths 63 ft, 46 ft and 40 ft, find the measure of the largest angle to the nearest degree.
5. Find the triangular area, to the nearest square meter, enclosed by three pieces of fencing 123 m, 150 m and 155 m long.
6. Find the area, to the nearest square inch, of a parallelogram with sides of length 12 in and 15 in and included angle of 78°.
7. A person at point A looks due east and spots a UFO with an angle of elevation of 40°. At the same time, another person, 1 mi due west of A looks due east and sights the same UFO with an angle of elevation of 25°. Find the distance between A and the UFO. How far is the UFO above the ground? Give answers to the nearest hundredth of a mile.
8. Find the area of a triangular playground, to the nearest square meter, with sides of length 10 m, 15 m and 16 m.

9. A yard is bounded on two sides with fences of length 80 ft and 60 ft. If these fences meet at a 75° angle, how many feet of fencing are required to completely enclosed a triangular region?
10. When a boy stands on the bank of a river and looks across to the other bank, the angle of depression is 12°. If he climbs to the top of a 10 ft tree and looks across to other bank, the angle of depression is 15°. What is the distance from the first position of the boy to the other bank of the river? How wide is the river? Give your answers to the nearest foot.

1.11 Chapter 8 Review

Keywords & Theorems

The Pythagorean Theorem

- Pythagorean Theorem
- Pythagorean Triple
- Distance Formula

The Pythagorean Theorem Converse

- Pythagorean Theorem Converse
- Theorem 8-3
- Theorem 8-4

Similar Right Triangles

- Theorem 8-5
- Geometric Mean

Special Right Triangles

- Isosceles Right (45-45-90) Triangle
- 30-60-90 Triangle
- 45-45-90 Theorem
- 30-60-90 Theorem

Tangent, Sine and Cosine Ratios

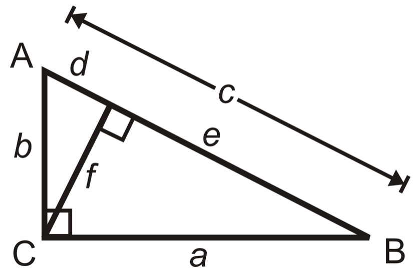
- Trigonometry
- Adjacent (Leg)
- Opposite (Leg)
- Sine Ratio
- Cosine Ratio
- Tangent Ratio
- Angle of Depression
- Angle of Elevation

Solving Right Triangles

- Inverse Tangent
- Inverse Sine
- Inverse Cosine

Review

Fill in the blanks using right triangle $\triangle ABC$.



1. $a^2 + \_\_\_\_^2 = c^2$
2. $\sin \_\_\_\_ = \frac{b}{c}$
3. $\tan \_\_\_\_ = \frac{f}{d}$
4. $\cos \_\_\_\_ = \frac{b}{c}$
5. $\tan^{-1}\left(\frac{f}{e}\right) = \_\_\_\_$
6. $\sin^{-1}\left(\frac{f}{b}\right) = \_\_\_\_$
7. $\_\_\_\_^2 + d^2 = b^2$
8. $\frac{?}{b} = \frac{b}{?}$
9. $\frac{e}{?} = \frac{?}{c}$
10. $\frac{d}{f} = \frac{f}{?}$

Solve the following right triangles using the Pythagorean Theorem, the trigonometric ratios, and the inverse trigonometric ratios. When possible, simplify the radical. If not, round all decimal answers to the nearest tenth.

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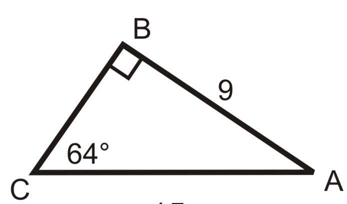

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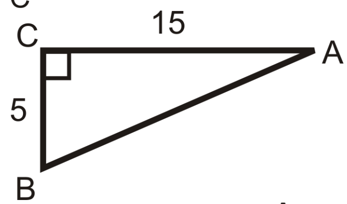

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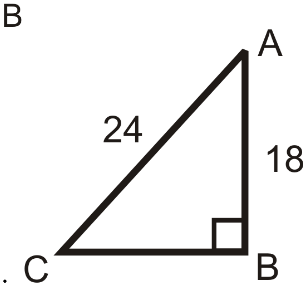
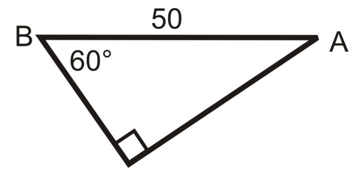
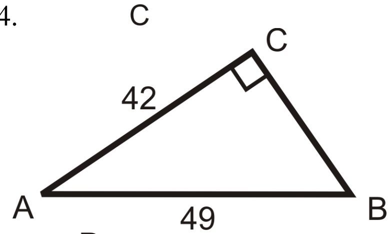
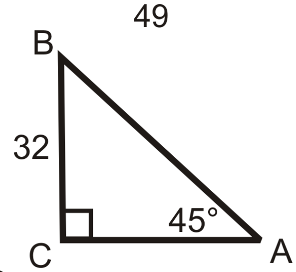
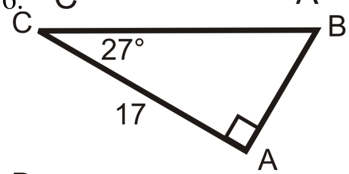
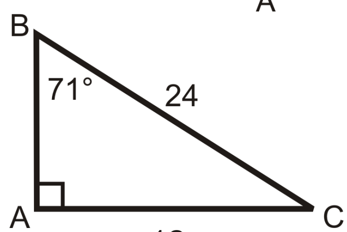


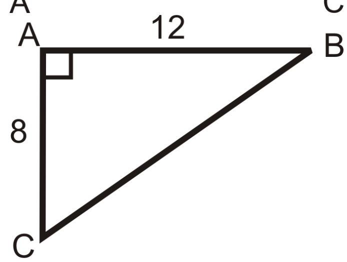
14. 

15. 

16. 

17. 

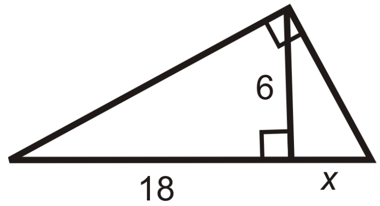
18. 

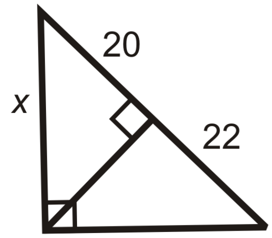
19. 

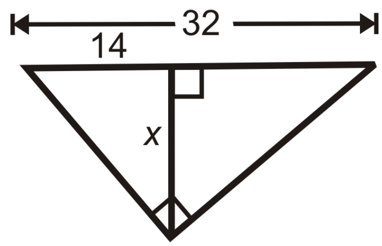
Determine if the following lengths make an acute, right, or obtuse triangle. If they make a right triangle, determine if the lengths are a Pythagorean triple.

20. 11, 12, 13
21. 16, 30, 34
22. 20, 25, 42
23. $10\sqrt{6}, 30, 10\sqrt{15}$
24. 22, 25, 31
25. 47, 27, 35

Find the value of x.

26. 

27. 

28. 

29. The angle of elevation from the base of a mountain to its peak is 76°. If its height is 2500 feet, what is the length to reach the top? Round the answer to the nearest tenth.
30. Taylor is taking an aerial tour of San Francisco in a helicopter. He spots ATT Park (baseball stadium) at a horizontal distance of 850 feet and down (vertical) 475 feet. What is the angle of depression from the helicopter to the park? Round the answer to the nearest tenth.

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9693 .

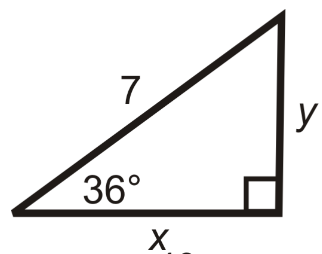
1.12 Inverse Trigonometric Ratios

Learning Objectives

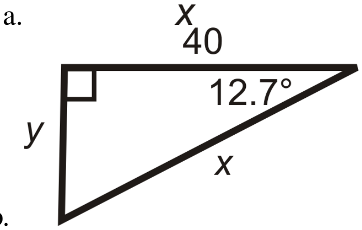
- Use the inverse trigonometric ratios to find an angle in a right triangle.
- Solve a right triangle.
- Apply inverse trigonometric ratios to real-life situation and special right triangles.

Review Queue

Find the lengths of the missing sides. Round your answer to the nearest hundredth.



a.



b.

c. Draw an isosceles right triangle with legs of length 3. What is the hypotenuse?
d. Use the triangle from #3, to find the sine, cosine, and tangent of 45°.
e. Explain why $\tan 45^\circ = 1$.

Know What? The longest escalator in North America is at the Wheaton Metro Station in Maryland. It is 230 feet long and is 115 ft high. What is the angle of elevation, x°, of this escalator?

Inverse Trigonometric Ratios

The word ***inverse*** is probably familiar to you. In mathematics, once you learn how to do an operation, you also learn how to "undo" it. For example, you may remember that addition and subtraction are considered inverse operations. Multiplication and division are also inverse operations. In algebra you used inverse operations to solve equations and inequalities.

When we apply the word inverse to the trigonometric ratios, we can find the acute angle measures within a right triangle. Normally, if you are given an angle and a side of a right triangle, you can find the other two sides, using sine, cosine or tangent. With the inverse trig ratios, you can find the angle measure, given two sides.

Inverse Tangent: If you know the opposite side and adjacent side of an angle in a right triangle, you can use inverse tangent to find the measure of the angle.

Inverse tangent is also called arctangent and is labeled $\tan^{-1}$ or *arctan*. The "-1" indicates inverse.

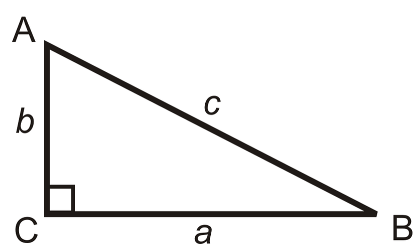
Inverse Sine: If you know the opposite side of an angle and the hypotenuse in a right triangle, you can use inverse sine to find the measure of the angle.

Inverse sine is also called arcsine and is labeled $\sin^{-1}$ or *arcsin*.

Inverse Cosine: If you know the adjacent side of an angle and the hypotenuse in a right triangle, you can use inverse cosine to find the measure of the angle.

Inverse cosine is also called arccosine and is labeled $\cos^{-1}$ or *arccos*.

Using the triangle below, the inverse trigonometric ratios look like this:



$$\tan^{-1}\left(\frac{b}{a}\right) = m\angle B \qquad \tan^{-1}\left(\frac{a}{b}\right) = m\angle A$$

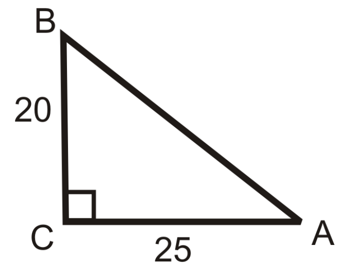
$$\sin^{-1}\left(\frac{b}{c}\right) = m\angle B \qquad \sin^{-1}\left(\frac{a}{c}\right) = m\angle A$$

$$\cos^{-1}\left(\frac{a}{c}\right) = m\angle B \qquad \cos^{-1}\left(\frac{b}{c}\right) = m\angle A$$

In order to actually find the measure of the angles, you will need you use your calculator. On most scientific and graphing calculators, the buttons look like $[\text{SIN}^{-1}], [\text{COS}^{-1}]$, and $[\text{TAN}^{-1}]$. Typically, you might have to hit a shift or 2^{nd} button to access these functions. For example, on the TI-83 and 84, $[2^{nd}][\text{SIN}]$ is $[\text{SIN}^{-1}]$. Again, make sure the mode is in degrees.

When you find the inverse of a trigonometric function, you put the word *arc* in front of it. So, the inverse of a tangent is called the arctangent (or arctan for short). Think of the arctangent as a tool you can use like any other inverse operation when solving a problem. If tangent tells you the ratio of the lengths of the sides opposite and adjacent to an angle, then tangent inverse tells you the measure of an angle with a given ratio.

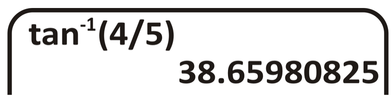
Example 1: Use the sides of the triangle and your calculator to find the value of $\angle A$. Round your answer to the nearest tenth of a degree.



Solution: In reference to $\angle A$, we are given the *opposite* leg and the *adjacent* leg. This means we should use the *tangent* ratio.

$\tan A = \frac{20}{25} = \frac{4}{5}$, therefore $\tan^{-1}\left(\frac{4}{5}\right) = m\angle A$. Use your calculator.

If you are using a TI-83 or 84, the keystrokes would be: $[2^{nd}]$[TAN] $\left(\frac{4}{5}\right)$ **[ENTER]** and the screen looks like:



So, $m\angle A = 38.7^\circ$

Example 2: $\angle A$ is an acute angle in a right triangle. Use your calculator to find $m\angle A$ to the nearest tenth of a degree.

a) $\sin A = 0.68$

b) $\cos A = 0.85$

c) $\tan A = 0.34$

Solution:

a) $m\angle A = \sin^{-1} 0.68 = 42.8^\circ$

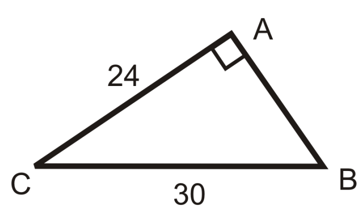
b) $m\angle A = \cos^{-1} 0.85 = 31.8^\circ$

c) $m\angle A = \tan^{-1} 0.34 = 18.8^\circ$

Solving Triangles

Now that we know how to use inverse trigonometric ratios to find the measure of the acute angles in a right triangle, we can solve right triangles. To solve a right triangle, you would need to find all sides and angles in a right triangle, using any method. When solving a right triangle, you could use sine, cosine or tangent, inverse sine, inverse cosine, or inverse tangent, or the Pythagorean Theorem. Remember when solving right triangles to only use the values that you are given.

Example 3: Solve the right triangle.



Solution: To solve this right triangle, we need to find $AB, m\angle C$ and $m\angle B$. Use AC and CB to give the most accurate answers.

$\underline{AB}$: Use the Pythagorean Theorem.

$$\begin{aligned}24^2 + AB^2 &= 30^2\\576 + AB^2 &= 900\\AB^2 &= 324\\AB &= \sqrt{324} = 18\end{aligned}$$

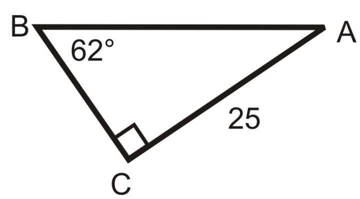
$\underline{m\angle B}$: Use the inverse sine ratio.

$$\begin{aligned}\sin B &= \frac{24}{30} = \frac{4}{5}\\\sin^{-1}\left(\frac{4}{5}\right) &= 53.1^\circ = m\angle B\end{aligned}$$

$\underline{m\angle C}$: Use the inverse cosine ratio.

$$\begin{aligned}\cos C &= \frac{24}{30} = \frac{4}{5}\\\cos^{-1}\left(\frac{4}{5}\right) &= 36.9^\circ = m\angle C\end{aligned}$$

Example 4: Solve the right triangle.



Solution: To solve this right triangle, we need to find AB, BC and $m\angle A$.

$\underline{AB}$: Use sine ratio.

$$\begin{aligned}\sin 62^\circ &= \frac{25}{AB}\\AB &= \frac{25}{\sin 62^\circ}\\AB &\approx 28.31\end{aligned}$$

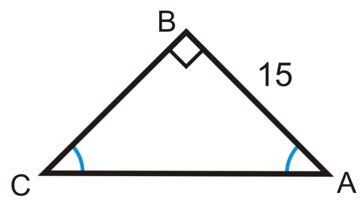
$\underline{BC}$: Use tangent ratio.

$$\begin{aligned}\tan 62^\circ &= \frac{25}{BC}\\BC &= \frac{25}{\tan 62^\circ}\\BC &\approx 13.30\end{aligned}$$

$\underline{m\angle A}$: Use Triangle Sum Theorem

$$\begin{aligned} 62^\circ + 90^\circ + m\angle A &= 180^\circ \\ m\angle A &= 28^\circ \end{aligned}$$

Example 5: Solve the right triangle.



Solution: Even though, there are no angle measures given, we know that the two acute angles are congruent, making them both 45°. Therefore, this is a 45-45-90 triangle. You can use the trigonometric ratios or the special right triangle ratios.

Trigonometric Ratios

$$\begin{aligned} \tan 45^\circ &= \frac{15}{BC} \\ BC &= \frac{15}{\tan 45^\circ} = 15 \end{aligned} \qquad \begin{aligned} \sin 45^\circ &= \frac{15}{AC} \\ AC &= \frac{15}{\sin 45^\circ} \approx 21.21 \end{aligned}$$

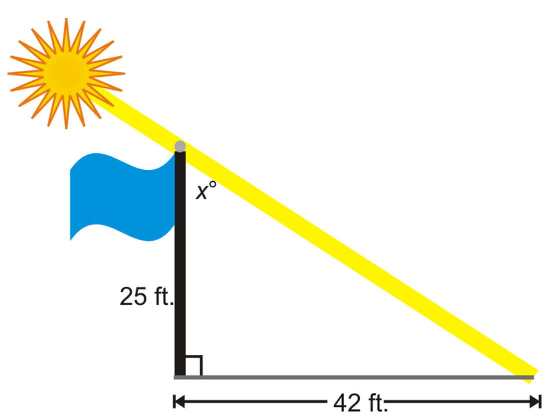
45-45-90 Triangle Ratios

$$BC = AB = 15, AC = 15\sqrt{2} \approx 21.21$$

Real-Life Situations

Much like the trigonometric ratios, the inverse trig ratios can be used in several real-life situations. Here are a couple examples.

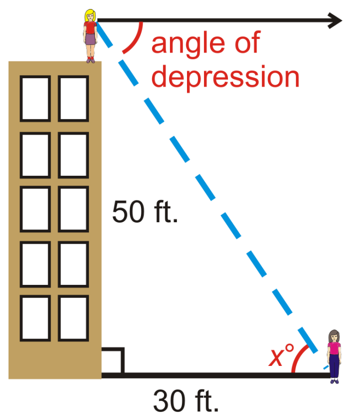
Example 6: A 25 foot tall flagpole casts a 42 feet shadow. What is the angle that the sun hits the flagpole?



Solution: First, draw a picture. The angle that the sun hits the flagpole is the acute angle at the top of the triangle, x°. From the picture, we can see that we need to use the inverse tangent ratio.

$$\tan x = \frac{42}{25}$$
$$\tan^{-1} \frac{42}{25} \approx 59.2^\circ = x$$

Example 7: Elise is standing on the top of a 50 foot building and spots her friend, Molly across the street. If Molly is 35 feet away from the base of the building, what is the angle of depression from Elise to Molly? Elise's eye height is 4.5 feet.



Solution: Because of parallel lines, the angle of depression is equal to the angle at Molly, or x°. We can use the inverse tangent ratio.

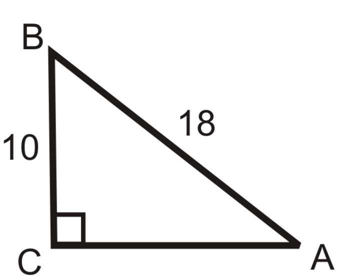
$$\tan^{-1}\left(\frac{54.5}{30}\right) = 61.2^\circ = x$$

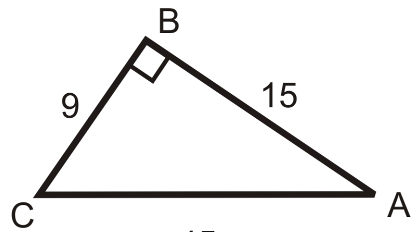
Know What? Revisited To find the escalator's angle of elevation, we need to use the inverse sine ratio.

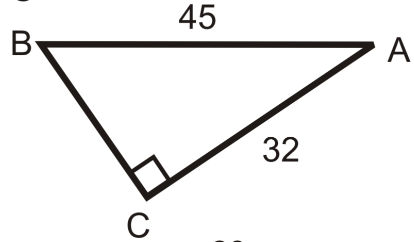
$$\sin^{-1}\left(\frac{115}{230}\right) = 30^\circ \qquad \text{The angle of elevation is } 30^\circ.$$

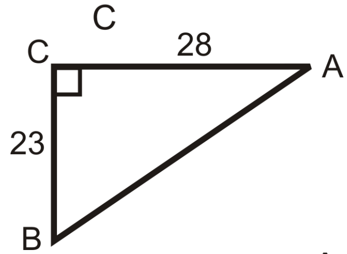
Review Questions

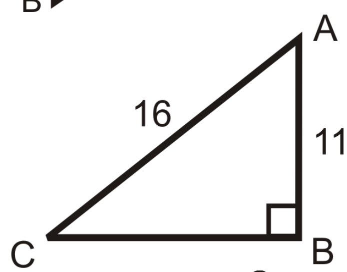
Use your calculator to find $m\angle A$ to the nearest tenth of a degree.

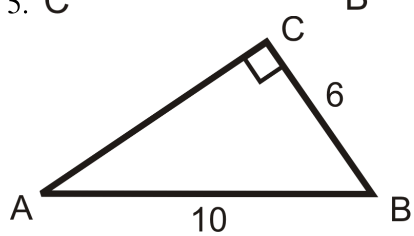
1. 

2. 

3. 

4. 

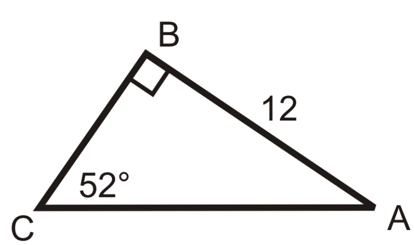
5. 

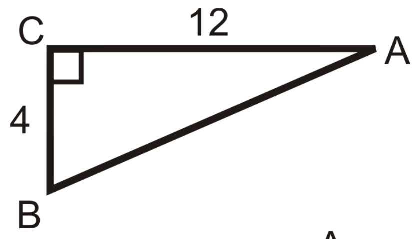
6. 

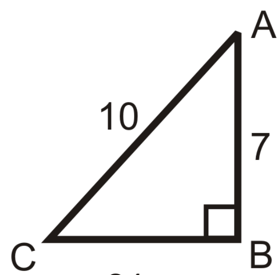
Let $\angle A$ be an acute angle in a right triangle. Find $m\angle A$ to the nearest tenth of a degree.

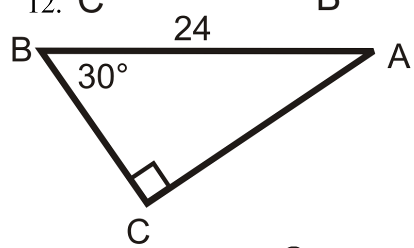
7. $\sin A = 0.5684$
8. $\cos A = 0.1234$
9. $\tan A = 2.78$

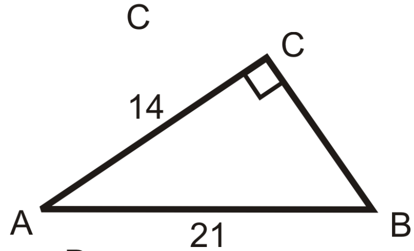
Solving the following right triangles. Find all missing sides and angles.

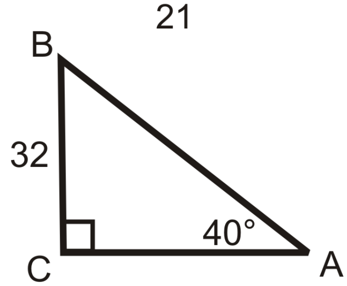
10. 

11. 

12. 

13. 

14. 

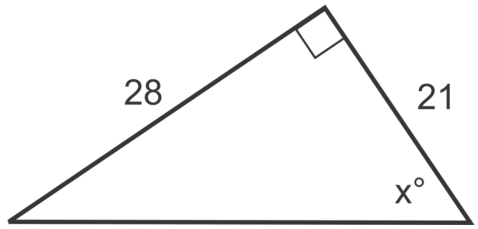
15. 

16. ***Writing*** Explain when to use a trigonometric ratio to find a side length of a right triangle and when to use the Pythagorean Theorem.

Real-Life Situations Use what you know about right triangles to solve for the missing angle. If needed, draw a picture. Round all answers to the nearest tenth of a degree.

17. A 75 foot building casts an 82 foot shadow. What is the angle that the sun hits the building?
18. Over 2 miles (horizontal), a road rises 300 feet (vertical). What is the angle of elevation?
19. A boat is sailing and spots a shipwreck 650 feet below the water. A diver jumps from the boat and swims 935 feet to reach the wreck. What is the angle of depression from the boat to the shipwreck?
20. Elizabeth wants to know the angle at which the sun hits a tree in her backyard at 3 pm. She finds that the length of the tree's shadow is 24 ft at 3 pm. At the same time of day, her shadow is 6 ft 5 inches. If Elizabeth is 4 ft 8 inches tall, find the height of the tree and hence the angle at which the sunlight hits the tree.
21. Alayna is trying to determine the angle at which to aim her sprinkler nozzle to water the top of a 5 ft bush in her yard. Assuming the water takes a straight path and the sprinkler is on the ground 4 ft from the tree, at what angle of inclination should she set it?
22. ***Science Connection*** Would the answer to number 20 be the same every day of the year? What factors would influence this answer? How about the answer to number 21? What factors might influence the path of the water?

23. Tommy was solving the triangle below and made a mistake. What did he do wrong?



$$\tan^{-1}\left(\frac{21}{28}\right) \approx 36.9^\circ$$

24. Tommy then continued the problem and set up the equation: $\cos 36.9^\circ = \frac{21}{h}$. By solving this equation he found that the hypotenuse was 26.3 units. Did he use the correct trigonometric ratio here? Is his answer correct? Why or why not?
25. How could Tommy have found the hypotenuse in the triangle another way and avoided making his mistake?

Examining Patterns Below is a table that shows the sine, cosine, and tangent values for eight different angle measures. Answer the following questions.

TABLE 1.5:

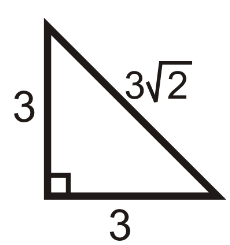
	10°	20°	30°	40°	50°	60°	70°	80°
Sine	0.1736	0.3420	0.5	0.6428	0.7660	0.8660	0.9397	0.9848
Cosine	0.9848	0.9397	0.8660	0.7660	0.6428	0.5	0.3420	0.1736
Tangent	0.1763	0.3640	0.5774	0.8391	1.1918	1.7321	2.7475	5.6713

26. What value is equal to $\sin 40^\circ$?
27. What value is equal to $\cos 70^\circ$?
28. Describe what happens to the sine values as the angle measures increase.
29. Describe what happens to the cosine values as the angle measures increase.
30. What two numbers are the sine and cosine values between?
31. Find $\tan 85^\circ, \tan 89^\circ$, and $\tan 89.5^\circ$ using your calculator. Now, describe what happens to the tangent values as the angle measures increase.
32. Explain why all of the sine and cosine values are less than one. (hint: think about the sides in the triangle and the relationships between their lengths)

Review Queue Answers

a. $\sin 36^\circ = \frac{y}{7}$ $\qquad$ $\cos 36^\circ = \frac{x}{7}$
$\quad y = 4.11$ $\qquad$ $x = 5.66$

b. $\cos 12.7^\circ = \frac{40}{x}$ $\qquad$ $\tan 12.7^\circ = \frac{y}{40}$
$\quad x = 41.00$ $\qquad$ $y = 9.01$

c. 

d. $\sin 45^\circ = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2}$
$\cos 45^\circ = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2}$
$\tan 45^\circ = \frac{3}{3} = 1$
e. The tangent of 45° equals one because it is the ratio of the opposite side over the adjacent side. In an isosceles right triangle, or 45-45-90 triangle, the opposite and adjacent sides are the same, making the ratio always 1.

1.13 Extension: Laws of Sines and Cosines

Learning Objectives

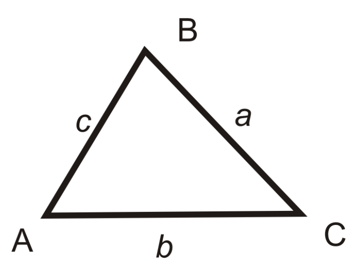
- Identify and use the Law of Sines and Cosines.

In this chapter, we have only applied the trigonometric ratios to right triangles. However, you can extend what we know about these ratios and derive the Law of Sines and the Law of Cosines. Both of these laws can be used with any type of triangle to find any angle or side within it. That means we can find the sine, cosine and tangent of angle that are greater than 90°, such as the obtuse angle in an obtuse triangle.

Law of Sines

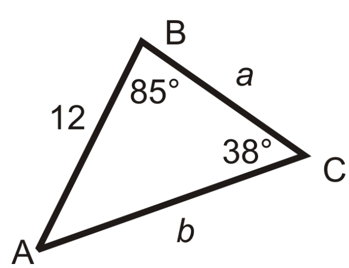
Law of Sines: If $\triangle ABC$ has sides of length, a, b, and c, then $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Looking at a triangle, the lengths a, b, and c are opposite the angles of the same letter. Let's use the Law of Sines on a couple of examples.



We will save the proof for a later course.

Example 1: Solve the triangle using the Law of Sines. Round decimal answers to the nearest tenth.



Solution: First, to find $m\angle A$, we can use the Triangle Sum Theorem.

$$\begin{aligned} m\angle A + 85^\circ + 38^\circ &= 180^\circ \\ m\angle A &= 57^\circ \end{aligned}$$

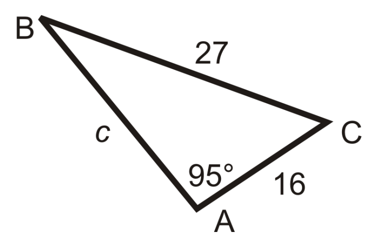
Now, use the Law of Sines to set up ratios for a and b.

$$\frac{\sin 57^\circ}{a}=\frac{\sin 85^\circ}{b}=\frac{\sin 38^\circ}{12}$$

$$\begin{aligned}\frac{\sin 57^\circ}{a}&=\frac{\sin 38^\circ}{12}\\ a\cdot\sin 38^\circ&=12\cdot\sin 57^\circ\\ a&=\frac{12\cdot\sin 57^\circ}{\sin 38^\circ}\approx 16.4\end{aligned}$$

$$\begin{aligned}\frac{\sin 85^\circ}{b}&=\frac{\sin 38^\circ}{12}\\ b\cdot\sin 38^\circ&=12\cdot\sin 85^\circ\\ b&=\frac{12\cdot\sin 85^\circ}{\sin 38^\circ}\approx 19.4\end{aligned}$$

Example 2: Solve the triangle using the Law of Sines. Round decimal answers to the nearest tenth.



Solution: Set up the ratio for $\angle B$ using Law of Sines.

$$\begin{aligned}\frac{\sin 95^\circ}{27}&=\frac{\sin B}{16}\\ 27\cdot\sin B&=16\cdot\sin 95^\circ\\ \sin B&=\frac{16\cdot\sin 95^\circ}{27}\to\sin^{-1}\left(\frac{16\cdot\sin 95^\circ}{27}\right)=36.2^\circ\end{aligned}$$

To find $m\angle C$ use the Triangle Sum Theorem. $m\angle C+95^\circ+36.2^\circ=180^\circ\to m\angle C=48.8^\circ$

To find c, use the Law of Sines again. $\frac{\sin 95^\circ}{27}=\frac{\sin 48.8^\circ}{c}$

$$\begin{aligned}c\cdot\sin 95^\circ&=27\cdot\sin 48.8^\circ\\ c&=\frac{27\cdot\sin 48.8^\circ}{\sin 95^\circ}\approx 20.4\end{aligned}$$

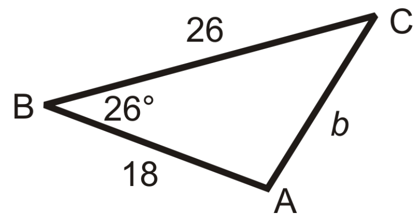
Law of Cosines

Law of Cosines: If $\triangle ABC$ has sides of length a,b, and c, then $a^2=b^2+c^2-2bc\cos A$

$$\begin{aligned}b^2&=a^2+c^2-2ac\ \cos B\\ c^2&=a^2+b^2-2ab\ \cos C\end{aligned}$$

Even though there are three formulas, they are all very similar. First, notice that whatever angle is in the cosine, the opposite side is on the other side of the equal sign.

Example 3: Solve the triangle using Law of Cosines. Round your answers to the nearest hundredth.



Solution: Use the second equation to solve for $\angle B$.

$$\begin{aligned}
b^2 &= 26^2 + 18^2 - 2(26)(18)\cos 26^\circ \\
b^2 &= 1000 - 936\cos 26^\circ \\
b^2 &= 158.7288 \\
b &\approx 12.60
\end{aligned}$$

To find $m\angle A$ or $m\angle C$, you can use either the Law of Sines or Law of Cosines. Let's use the Law of Sines.

$$\frac{\sin 26^\circ}{12.60} = \frac{\sin A}{18}$$

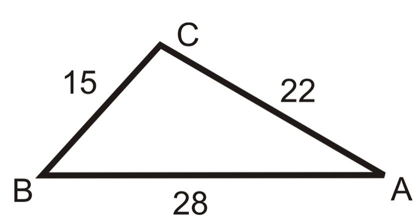
$$\begin{aligned}
12.60 \cdot \sin A &= 18 \cdot \sin 26^\circ \\
\sin A &= \frac{18 \cdot \sin 26^\circ}{12.60}
\end{aligned}$$

$\sin^{-1}\left(\frac{18 \cdot \sin 26^\circ}{12.60}\right) \approx 38.77^\circ$ To find $m\angle C$, use the Triangle Sum Theorem.

$$\begin{aligned}
26^\circ + 38.77^\circ + m\angle C &= 180^\circ \\
m\angle C &= 115.23^\circ
\end{aligned}$$

Unlike the previous sections in this chapter, with the Laws of Sines and Cosines, we have been using values that we have found to find other values. With these two laws, you have to use values that are not given. Just keep in mind to always wait until the very last step to put anything into your calculator. This will ensure that you have the most accurate answer.

Example 4: Solve the triangle. Round your answers to the nearest hundredth.



Solution: When you are given only the sides, you have to use the Law of Cosines to find one angle and then you can use the Law of Sines to find another.

$$\begin{aligned}15^2 &= 22^2 + 28^2 - 2(22)(28)\cos A\\ 225 &= 1268 - 1232\cos A\\ -1043 &= -1232\cos A\\ \frac{-1043}{-1232} &= \cos A \rightarrow \cos^{-1}\left(\frac{1043}{1232}\right) \approx 32.16^\circ\end{aligned}$$

Now that we have an angle and its opposite side, we can use the Law of Sines.

$$\frac{\sin 32.16^\circ}{15} = \frac{\sin B}{22}$$

$$\begin{aligned}15 \cdot \sin B &= 22 \cdot \sin 32.16^\circ\\ \sin B &= \frac{22 \cdot \sin 32.16^\circ}{15}\end{aligned}$$

$\sin^{-1}\left(\frac{22 \cdot \sin 32.16^\circ}{15}\right) \approx 51.32^\circ$ To find $m\angle C$, use the Triangle Sum Theorem.

$$\begin{aligned}32.16^\circ + 51.32^\circ + m\angle C &= 180^\circ\\ m\angle C &= 96.52^\circ\end{aligned}$$

To Summarize

Use Law of Sines when given:

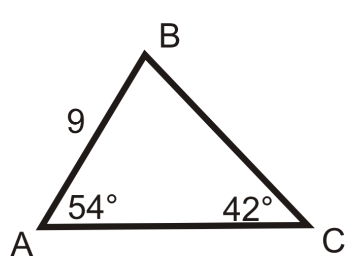
- An angle and its opposite side.
- Any two angles and one side.
- Two sides and the non-included angle.

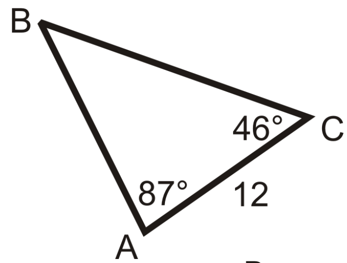
Use Law of Cosines when given:

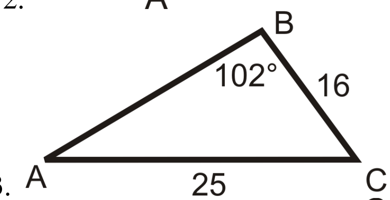
- Two sides and the included angle.
- All three sides.

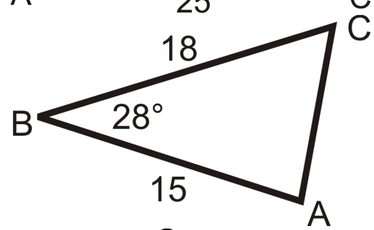
Review Questions

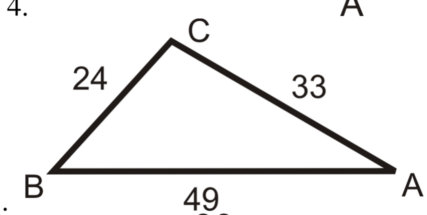
Use the Law of Sines or Cosines to solve $\triangle ABC$. If you are not given a picture, draw one. Round all decimal answers to the nearest tenth.

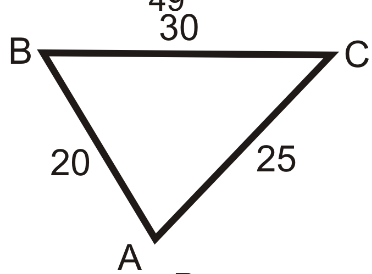
1. 

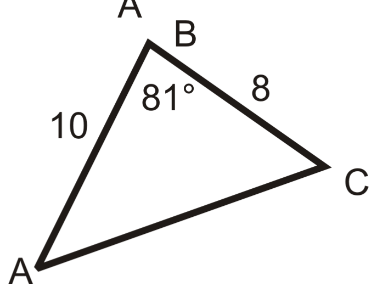
2. 

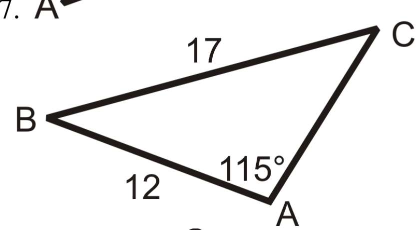
3. 

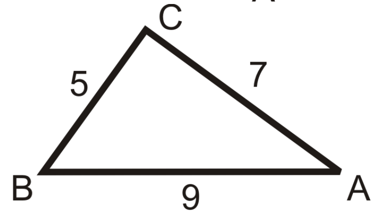
4. 

5. 

6. 

7. 

8. 

9. 

10. $m\angle A = 74^\circ, m\angle B = 11^\circ, BC = 16$

11. $m\angle A = 64^\circ, AB = 29, AC = 34$
12. $m\angle C = 133^\circ, m\angle B = 25^\circ, AB = 48$

Use the Law of Sines to solve $\triangle ABC$ below.

13. $m\angle A = 20^\circ, AB = 12, BC = 5$

Recall that when we learned how to prove that triangles were congruent we determined that SSA (two sides and an angle not included) did not determine a unique triangle. When we are using the Law of Sines to solve a triangle and we are given two sides and the angle not included, we may have two possible triangles. Problem 14 illustrates this.

14. Let's say we have $\triangle ABC$ as we did in problem 13. In problem 13 you were given two sides and the not included angle. This time, you have two angles and the side between them (ASA). Solve the triangle given that $m\angle A = 20^\circ, m\angle C = 125^\circ, AC = 8.4$
15. Does the triangle that you found in problem 14 meet the requirements of the given information in problem 13? How are the two different $m\angle C$ related? Draw the two possible triangles overlapping to visualize this relationship.

It is beyond the scope of this text to determine when there will be two possible triangles, but the concept of the possibility is something worth noting at this time.